Urban
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Ranks of Selmer Groups Part 1:

E/K elliptic curve, $K = \#$ field p prime.

$$0 \to E[p^n] \to E \xrightarrow{p^n} E \to 0$$

Take Galois cohom.

$$0 \to E(K) \otimes \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\kappa} H^1(K, E[p^n]) \to H^1(K, E)_{p^n} \to 0$$

$$0 \to E(K_v) \otimes \mathbb{Z}/p^n\mathbb{Z} \to H^1(K_v, E[p^n]) \to H^1(K_v, E)_{p^n} \to 0$$

Take inductive limit:

$$\begin{array}{ccccccccc} 0 \to & E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \to & Sel(K,E)_p & \to & Ш(K,E)_p \to 0 \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p & \to & H^1(K, T_p(E) \otimes \mathbb{Q}_p/\mathbb{Z}_p) & \to & H^1(K,E)_{p^\infty} \to 0. \\ & & & \downarrow & & \\ & 0 & \to & \bigoplus_v \dfrac{H^1(K_v, T_pE \otimes \mathbb{Q}_p/\mathbb{Z}_p)}{\operatorname{Im} \kappa_v} & \to & \bigoplus_v H^1(K_v, E)_{p^\infty} \to 0 \end{array}$$

$K = \mathbb{Q}$:

BSD: weak version $\operatorname{ord} L(E,s)|_{s=1} = \underbrace{\operatorname{rk} E(K) = \operatorname{corank} Sel(K,E)}_{\text{easier to study.}}$

Study the corank of $Sel(K,E)$.

For this lecture $K = \mathbb{Q}$. or imag. quad field. , $E/\mathbb{Q}$ always.

Greenberg, Mazur, Bloch-Kato have defined other Selmer groups, i.e. subgroups of $H^1(K, V_p(E))$ (where $V_p(E) = T_p(E) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$)
that are the same rank as $Sel_p(K,E)$. We denote these

by $H^1_f(K, V_p(E))$. $H^1(K, V_p(E))$ classifies extensions of G_K-reps : ($V = V_p(E)/L$, L = fin. ext. of $\mathbb{Q}_p$)

$$(*) \quad 0 \to V \to W \to L \to 0$$

The group $H^1_f(K, V_p(E))$ classifies extensions satisfying:

v finite place, $v \nmid p$: $0 \to V^{I_{K_v}} \to W^{I_{K_v}} \to L \to 0$

for $v \mid p$ if V crystalline, W should be crystalline.

Bloch-Kato def of H^1_f.

Given $(*)$, one gets H^1 by taking G_K-invariants.

If V is ordinary at $v \mid p$, we have a filtration $Fil^i V \subset V$, $gr^i V \cong \varepsilon^i$, $i \in \mathbb{Z}$.

$$0 \to V/Fil^1 V \to W/Fil^1 V \to L \to 0$$

$[W] \in H^1_{green}(K, V)$ if image of $[W]$ in $H^1(K_v, V/Fil^1 V)$ is 0 $\forall v \mid p$.

<u>Remark:</u> $H^1_f \subset H^1_g \subset H$

$\uparrow$
require deRham
$\Rightarrow$ deRham

$H^1_f = H^1_g$ in the case that q_v^{-1} is not an eigenvalue of Φ acting on $D_{st}(V)$.
(using geometric convention, i.e., H-T weight of $\varepsilon = -1$).

If $V = V_p(E)$ and if E has good reduction at $v \mid p$,

$$H^1_f(K, V_p(E)) = H^1_g(K, V_p(E)).$$

f cuspidal eigenform of even wt $2m$ for $\Gamma_0(N)$. Let E_f = field generated by evs. $\mathfrak{p}|p$ in E_f, $L = E_{\mathfrak{p}}$. There is a Galois rep.

$$V_f \circlearrowleft G_{\mathbb{Q}} \qquad \text{H-T-weights } (0, 2m-1)$$

attached to f. 2-dim $/L$. Since we use geo. conventions, $\det(V_f) = \varepsilon_{cyc}^{1-2m}$.

We will consider $\overset{V}{\overset{\shortparallel}{V_f(m)}}$. This is pure of wt 1. This has H-T-weights $(-m, m-1)$. Further, $V^\vee(1) \simeq V$.

__Conjecture (Bloch-Kato, $BSD_{k=2}$):__

$$\operatorname{ord} L(f,s)\big|_{s=m} = \operatorname{rk}_L H^1_f(\mathbb{Q}, V_f(m)).$$

__Thm (S-U):__ Same hypotheses. Assume $p \nmid N$, $m \geq 1$.

1. Then if $L(f,m) = 0 \Rightarrow \operatorname{rk} H^1_f \geq 1$.
2. If moreover $\operatorname{ord} L(f,s)|_{s=m}$ is even $(\varepsilon(f,m) = +1)$ then $\operatorname{rk} H^1_f \geq 2$. (under some technical conds on Gal. reps)

__Historical remarks:__

- wt $2m=2$ BSD is known if $\operatorname{ord} L(f,s)|_{s=1} = 0$ or 1.

  (Gross-Zagier, Kolyvagin, Rubin)

	$\varepsilon = -1$	$\varepsilon = 1$
$m > 1$	• Saito-Kurokawa lift when f ord. (extend to crystalline case) • MC [only rk ≥ 1 in this box]	ICM notes rk ≥ 2
$m = 1$	• MC ord. crystalline case new result. Can extend SK-method [only rk ≥ 1 this box]	New result, extension of the method $m > 1$.

Deformations of reducible Galois representations:

Γ grp, $\mathcal{O}/\mathbb{Z}_p$, A an $\mathcal{O}$-alg. (topologically f.g.) (int. domain)

$A_L = A \otimes_{\mathcal{O}} L$ $\quad L = \mathrm{Frac}(\mathcal{O})$.

$\mathfrak{m} \subset A$ maximal ideal, $A/\mathfrak{m} = k$

$\Gamma \xrightarrow{T} A$, T a pseudo-representation of dim n. $\left(\begin{array}{l}\text{think trace of a rep.)} \\ T = \mathrm{Tr}(r), \\ r : \Gamma \to GL_n(\mathrm{Frac}(A))\end{array}\right)$

Assume:

1) r is absolutely irred.
2) $\mathrm{tr}(r(\gamma)) \in A \quad \forall \gamma \in \Gamma$.
3) $\mathrm{tr}(r) \bmod \mathfrak{m} = \mathrm{tr}(\bar\rho_1(\gamma)) + \cdots + \mathrm{tr}(\bar\rho_s(\gamma))$ where $\bar\rho_i : \Gamma \to GL_{n_i}(A/\mathfrak{m})$ abs. irred.

Further, assume $\forall i \neq s$, $\bar\rho_i \not\cong \bar\rho_s$.

$$A[\Gamma] \longrightarrow \mathrm{Mat}_{n_1}(k) \times \cdots \times \mathrm{Mat}_{n_{s-1}}(k) \times \mathrm{Mat}_{n_s}(k)$$

~~Usually~~ If one assume $\bar\rho_i \not\cong \bar\rho_j$, this map is surjective. Under this weaker hypothesis here, one gets $(0, \ldots, 0, 1)$ in the image.

One can lift 0×1 to an idempotent of $A_{\mathfrak{m}}[\Gamma]$,

e_s = image of this idempotent under r.

Choose $v \in \mathrm{Frac}(A)^n$, $e_s v = v$, $\mathcal{L}_{\rho_s} = A_{\mathfrak{m}}[\Gamma]$-stable lattice generated by v.

Lemma: $\mathcal{L}_{\rho_s}$ is a Γ-stable lattice of the representation r and it has a unique irred. qt (up to isom.) and this quotient is isomorphic to $\bar{\rho}_s$.

Assume that $\mathcal{L}_{\rho_s}$ is free.

$$\bar{\mathcal{L}} = \mathcal{L}_{\rho_s}/m\mathcal{L}_{\rho_s} \longrightarrow \bar{\rho}_s \longrightarrow 0$$

$$\bar{\mathcal{L}}^{ss} \simeq \oplus \bar{\rho}_i .$$

Since $\bar{\mathcal{L}}$ is not semi-simple, we can consider a filtration:

$$\underset{i \neq s}{0 \longrightarrow \bar{\rho}_i} \longrightarrow \bar{\mathcal{L}}/\bar{\mathcal{L}}^{(1)} \longrightarrow \bar{\rho}_s \longrightarrow 0 \quad \rightsquigarrow \operatorname{Ext}^1_{\Gamma}(\bar{\rho}_s, \bar{\rho}_i)$$

However, need to have:

- be able to choose the i you want (gives problem for tot. real fields)
- prove the local properties of the extension when $\Gamma = G_{\mathbb{Q}}$.

Use the theory of finite slope families of rep. of $G_{\mathbb{Q}_p}$.
(triangulane rep.)
Ordinary case: Hida
Positive slope: Kisin, Colmez

$A = A(X)$ $x \in X/\mathbb{Q}_p$ affinoid, $X = X(\bar{\mathbb{Q}}_p)$.

$$G_{\mathbb{Q}_p} \xrightarrow{T} A(X) \xrightarrow{ev_x} L_x \subset \bar{\mathbb{Q}}_p .$$

$T_x = ev_x \circ T = tr(\rho_x)$, $\rho_x : G_{\mathbb{Q}_p} \longrightarrow GL_n(L_x)$

$\kappa_1, \dots, \kappa_n \in A(X)$ s.t. $\forall x \in X$ $\kappa_1(x), \dots, \kappa_n(x)$ $\kappa_1(x) \leq \kappa_2(x) \leq \dots \leq \kappa_n(x)$

are the Hodge-Tate ~~Sen~~ weights of ρ_x.

$\varphi_1, \varphi_2, \ldots, \varphi_n \in A(X)$

$\Sigma \subset X$ Zariski dense and s.t. $\forall x \in \Sigma$, ρ_x crystalline with Frobenius eigenvalues $\varphi_1(x) p^{\kappa_1(x)}, \ldots, \varphi_m(x) p^{\kappa_m(x)}$.

Remark: 1) slope $\doteq (v(\varphi_1(x)), \ldots, v(\varphi_m(x)))$

2) If slope $= (0, \ldots, 0)$ the family is ordinary

~~assume~~ ~~assume~~ this

$i_0 \geq 1$ s.t. $\underset{i \in [1, i_0]}{\kappa_i(x) - \kappa_1(x)}$ is bounded $\forall x \in X$.

$\forall i > i_0$, $\{\kappa_i(x) - \kappa_1(x) > N, x \in \Sigma\} = \Sigma_N$ is Zariski dense $\forall N$.

Prop. (Kisin): $\mathcal{L} \subset \mathrm{Frac}(X)$ free lattice over $A(X)$. Then

$$D_{cris}(\mathcal{L} \otimes L_x)^{\varphi = \varphi_i(x) p^{\kappa_i(x)}} \neq 0$$

$\forall i \in \{1, \ldots, i_0\}$.

Can use this to prove exts are deRham.

Global deformations:

K imag. quad. field

V p-adic rep. of G_K

$V^\vee(1) \simeq V^c$ attached to auto. rep. of some unitary group.

$n = 1$ $V =$ anticyclotomic char.

$$L(V, s) = \varepsilon(V, s) L(V^\vee(1), -s) = \varepsilon(V, s) L(V, -s).$$

$$\left.\begin{array}{c} \mathrm{ord}\left(L(V,s)\big|_{s=0} \right) \geq 1 \\ \uparrow \\ V \text{ attached to some} \\ \text{cusp rep. of some unitary} \\ \text{grp. } U(a,b) \end{array}\right\} \Rightarrow \begin{array}{c} \text{ICM paper} \\ \downarrow \\ \text{Eisenstein series} \\ U(a+1, b+1) \end{array}$$

$R(\text{Eis. series}) \cong V \oplus L \oplus L(1)$.

Get a deformation of Galois representations

Suppose $V = V_{f(m)}\big|_{G_K} \longrightarrow H^1_f(K, V^\vee(1))$.

$\mathcal{M} = \ker(ev_{x_0})$

$T : G_K \to A(x)$ $\qquad X \ni x_0$ attached to $\rho^{ss}_{x_0} \cong V \oplus L \oplus L(1)$

$\mathcal{L}$ free lattice with unique quotient $\simeq V$.

$$\begin{pmatrix} \varepsilon & * & * \\ 0 & 1 & * \\ 0 & 0 & \rho_V \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * \\ 0 & \varepsilon & * \\ 0 & 0 & \rho_V \end{pmatrix}$$

Need to rule out this case.

happy for this one get ext. we want and it is deRham by Kisin

Unram. $\forall v \nmid p$. if $m > 1$, $\Rightarrow *$ is crystalline

$\Rightarrow$ get ext. in $H^1_f(K, \mathbb{Q}_p(1))$,

which has rank # rts of unity,

there are none. $\Rightarrow$ this case is ruled-out.

If $m=1$, can't apply Kisin.

$\begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$ is semi-stable $\rightsquigarrow$ $N \hookrightarrow D_{st}\left(\begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}\right)$

Take Λ^2 of $\begin{pmatrix} \varepsilon & * & * \\ 0 & 1 & * \\ 0 & 0 & \rho_r \end{pmatrix}$, get 6-dim rep. and get

$V \otimes \begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}$ as a subqt.

$$D_{st}\left(V \otimes \begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}\right) = D_{st}(V) \otimes D_{st}\left(\begin{pmatrix} \varepsilon & * \\ 0 & 1 \end{pmatrix}\right). \quad Id \otimes N,$$

Can now apply Kisin to get $Id \otimes N = 0 \Rightarrow N = 0$ and we are done. This gives the rk 1 case.