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Ranks of Selmer Groups Part II:

K imag. quadratic

V p-adic rep. of G_K, irred., cont., fin. dim, motivic, pure,

$V^c \cong V^\vee(1)$ (c = nontrivial auto. of K, $V^c = V \circ c$).

Would like to show $L(V,0) = 0 \Rightarrow H^1_f(K, V^\vee(1)) \neq 0$.

We consider deformations of $\varepsilon \oplus V \oplus 1$. (Would like to look at $\varepsilon \oplus V$, but need to throw on the 1 so the rep. satisfies the self-dual condition given above.)

Today will explain the connection of $L(V,0) = 0$ giving deformations of $\varepsilon \oplus V \oplus 1$ for V cuspidal auto. There are two parts to this:

1) $L(V,0) = 0 \Rightarrow$ existence of a (p-adically) holomorphic modular form with associated Galois rep. $\varepsilon \oplus V \oplus 1$.

2) deforming these to "special" modular forms.

Focus on 3 examples:

(a) $\zeta(-2m) = 0$ if $m > 1$ and $\zeta(0) \neq 0$.

(b) $V = V_f(k)$ f = wt $2k$, trivial char., g newform

$L(V,0) = L(f,k)$, $\varepsilon(f,k) = -1$.

(c) $\varepsilon(f,k) = \pm 1$.

For odd weight, there are no Selmer groups expected to be infinite.

(a) $$G_{2k}(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} j(\gamma,\tau)^{-2k} |j(\gamma,\tau)|^{-s} \qquad Re(s+2k) >> 0$$

$$= \sum_{m=-\infty}^{\infty} c_m(s,y)\, e(mx) \qquad (\tau = x+iy)$$

In fact, one can compute

$$c_0(s,y) = (1+(-1)^k \frac{\Gamma(1-\frac{s+2k}{2})\Gamma(\frac{s+2k}{2})\zeta(2-s-2k)}{\sqrt{\pi}\,\Gamma(s/2)\Gamma(s/2+2k)\Gamma(\frac{1-s-2k}{2})\zeta(1-s-2k)}\, y^{1-2k-s}$$

This is analytic at $s=0$ $\forall\ 2k \geq 2$.

Holomorphic in τ at $s=0 \Leftrightarrow \zeta(2-2k) \cancel{\neq} = 0$. Thus, $G_{2k}(\tau,0)$ is holomorphic in $\tau \Leftrightarrow \zeta(2-2k) = 0$ iff $2k \geq 4$.

(One could actually use that $H_f^1(\mathbb{Q}, \mathbb{Q}_p(1)) = 0$ to show $\zeta(0) = 0$!)

(b) f wt $2k$, trivial char, $\varepsilon(f,k) = -1 \Rightarrow L(V,0) = 0$.

In this case, $\exists$ a Siegel modular form F of wt $k-1$ that is (NOTE: Chris used wt k, but S-K lift goes $2k \to k-1 \ldots$)

- cuspidal
- holomorphic
- $L(\mathrm{spin}, F, s) = L(f,s)\zeta(s-k+1)\zeta(s-k+2)$ CAP form
- $\pi \leftrightarrow F$

π_∞ holo. discrete series

$\otimes \pi_v$ "minimally ramified"

The factorization of the L-fctn gives

$$\rho_F \simeq \rho_f \oplus \varepsilon^{1-k} \oplus \varepsilon^{2-k}$$

Thus,

$$\rho_F(k) \simeq V \oplus \varepsilon \oplus \varepsilon^2.$$

[Can shift all this to agree with what Chris presented in the lecture. I may be missing why it would be wt k...]

(c) Let

$$T_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \in GL_{2n}$$

G_n = unitary group assoc. to skew Hermitian pair (K^{2n}, T_n).

R = $\mathbb{Q}$-alg.

$G_n(R) = \{ g \in GL_{2n}(K \otimes_{\mathbb{Q}} R) : g T_n {}^t\bar{g} = T_n \}$

$G_n(\mathbb{R}) \simeq U(n,n)$

P_n = stab. of $\underbrace{0 \oplus \cdots \oplus 0}_{2n-1} \oplus K$ $\qquad$ P_n = max $\mathbb{C}$-parabolic $= M_n N_n$.

$$M_n = \left\{ \begin{pmatrix} A & & B & \\ & \bar{t}^{-1} & & \\ C & & D & \\ & & & t \end{pmatrix} : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{n-1}, \quad t \in \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \right\}$$

$$\simeq G_{n-1} \times \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$$

N_n = unipotent radical.

Eisenstein series:

π cusp. rep. of $G_{n-1}(\mathbb{A})$, χ idele class char. of $\mathbb{A}_K^\times = (\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m)(\mathbb{A})$.

$\rho = \rho_{\pi,\chi}$ = rep. of $P_n(\mathbb{A})$ on V_π

$$\rho(m(g,t)n)v = \pi(g)\chi(t)v$$

δ_n = modulus character for P_n. $\quad \delta_n(m(g,t)) = |t\bar{t}|^{-(2n-1)}$.

$I(\rho) \ni \varphi$

$$E(\varphi, g, s) = \sum_{\gamma \in P_n(\mathbb{Q}) \backslash G_n(\mathbb{Q})} \varphi(\gamma g)\, \delta(\gamma g)^{s+1/2} \qquad \text{converges for } \mathrm{Re}(s) >> 0.$$

$g \in G_n(\mathbb{A})$, $s \in \mathbb{C}$

(Note $\varphi(\gamma g)$ really takes values in V_π, so we always evaluate at 1 when defining a function.)

The interesting info. is in the constant terms. The only

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interesting one is along P_n:

$$E_{P_n}(\varphi, g, s) = \varphi(g)\delta(g)^{s+1/2} + M(s)(\varphi)(s)\delta(g)^{-s+1/2}$$

$\varphi = \otimes \varphi_v$, $I(\rho) = \otimes I(\rho_v)$.

$$M(s)(\varphi) = \otimes \underbrace{M_v(s)(\varphi_v)}$$

for a.e. place one has this is $\overbrace{\dfrac{L(\pi_v, \chi_v^{-1}, (2n-1)s)\, L(\chi_v^{-1}|_{\mathbb{Q}_v^\times}, 2(2n-1)s)}{L(\pi_v, \chi_v^{-1}, (2n-1)s+1)\, L(\chi_v^{-1}|_{\mathbb{Q}_v^\times}, 2(2n-1)s+1)}}^{(*)}$

The only way for $(*)$ to give a pole is if it contains the Riemann zeta function.

Let $H = G_2$, $G = G_1$. f wt $2k$, trivial char, level N.
Associate to f the form $\varphi(g) = j(g_\infty, i)^{-2k} f(g_\infty(i))$, $g = \gamma g_\infty k$
$$\in GL_2(\mathbb{A}) = GL_2(\mathbb{Q}) GL_2^+(\mathbb{R}) K_0(N).$$

ψ idele class char. of $\mathbb{A}_K^\times$ s.t. $\psi_\infty(z) = z^{-2k}$,
$\psi|_{\mathbb{A}^\times} = |\cdot|^{-2k}$

$$\mathbb{A}_K^\times GL_2(\mathbb{A}) = GU(1,1)_{\mathbb{A}} \overset{?}{=} G(\mathbb{A}).$$
$$\wr\!\mid$$
$$GL_2(\mathbb{A}_K)$$

Write $g = tx$ $\quad \varphi_\psi(g) = \psi(t)\varphi(x) \rightsquigarrow \pi$ on $G(\mathbb{A})$.
$\chi = \psi^{-c} =_s \psi^{-1} \circ c$

This info. is used to define the Eisenstein series.

We want to get $L(v, s)\zeta(s+1)\zeta(s)$ up to twists.

$$V \oplus \varepsilon \oplus 1$$

HT-wts $\quad (-k, \overset{k-1}{1+k}) \quad -1 \quad 0$

This H-T wts predict the wt of the E.S. we want.

$(2-k, 0, 4, k+2)$. $\left(\begin{matrix}\text{Need } k+2 \geq 4 \\ \Rightarrow 2k \geq 4\end{matrix}\right)$.

This tells where to look for the Eisenstein series.

Back to the intertwining operators: In this case we get:

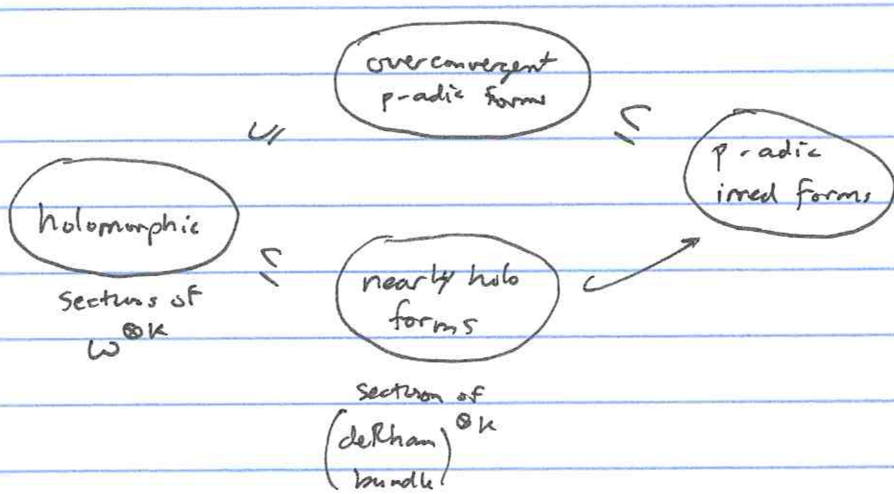
$$3s_0 = 1/2 + k$$

$$L(f, \psi^{-c}\overset{1}{\cancel{\chi^{-1}}}, k)\ \zeta(1) = \text{numerator.. at } s_0 = \tfrac{1}{6} + k/3.$$

This only works for $2k \geq 4$, so excludes elliptic curves.

If sign of f.e. is -1, can use p-adic family to get down to wt 2.

If sign is $+1$, get nearly holo E.S.



$f = f_2 \qquad \{f_{2k}\}$ p-adic family

$\underset{k>1}{f_{2k}} \rightsquigarrow \left\{\begin{matrix}\text{nearly holo} \\ \text{E.S.}\end{matrix}\right\}$ p-adic family

$L(f_2, 1) = 0 \rightsquigarrow$ should deform into family of holo. things.