Introduction to Shimura varieties:

Motivation is modular curves, Hilbert modular varieties, Siegel modular varieties. These are all special cases of Shimura varieties.

1. Shimura datum - input

(G, X)

G = connected reductive group over $\mathbb{Q}$

X = finite disjoint union of Hermitian symmetric domains.

$\left(\simeq G(\mathbb{R}) / K'_\infty \right)$ ← almost max. compct subgrp.

X is the $G(\mathbb{R})$ - conjugacy classes of

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}}.$$

We don't allow any (G, X), they are subject to a list of axioms. (ex: $G \neq GL_n$, $n \geq 3$).

Example: $n \geq 1$, $J = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \in \mathrm{Mat}_{2n}(\mathbb{Z})$

$$G = GSp_{2n} = \{ (g, \lambda) \in GL_{2n} \times \mathbb{G}_m : {}^t g J g = \lambda J \}.$$

$$X = \mathfrak{H}_n^{\pm} = \{ z \in \mathrm{Mat}_n(\mathbb{C}) : {}^t z = z, \ \mathrm{Im}(z) > 0 \text{ or } \mathrm{Im}(z) < 0 \}.$$

Specialize to $n = 1$:

$$\left. \begin{array}{l} GSp_2 = GL_2 \\ X = \mathfrak{H}^{\pm} = \mathbb{C} - \mathbb{R}. \end{array} \right\} \rightsquigarrow \text{gives rise to modular curves.}$$

$$\begin{array}{ccc} \mathfrak{H}^{\pm} & & \\ \text{IS} & & \\ GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \mathbb{R}^\times & & \mathbb{C} - \mathbb{R} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \dfrac{ai + b}{ci + d}. \end{array}$$

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<u>Example:</u> G = torus

X = pt.

2. Overview:

Shimura datum (G, X) gives rise to

(i) a reflex field $E(G,X) \subset \mathbb{C}$ $\left(G = GSp_{2n} \Rightarrow E(G,X) = \mathbb{Q} \right)$

$\uparrow$ # field

(ii) Shimura variety $\quad \swarrow$ finite adeles

$Sh(G,X) \circlearrowleft G(\mathbb{A}^\infty)$ over $E(G,X)$.

"Hecke action"

$$Sh(G,X) = \left\{ \begin{array}{c} \text{projective system} \\ \\ \searrow \quad \downarrow \quad \swarrow \\ \\ Sh_{K'}(G,X) \\ \downarrow \quad \swarrow \\ Sh_K(G,X) \end{array} \right\}$$

$K \leq G(\mathbb{A}^\infty)$ open compact subgroups.

- $\forall g \in G(\mathbb{A}^\infty)$, $\exists$ canonical map
$$g: Sh_{K'}(G,X) \to Sh_K(G,X)$$
if $g^{-1}K'g \leq K$.

Each $Sh_K(G,X)$ is a quasi-projective smooth variety over $E(G,X)$ (possibly disconnected)

$Sh(G,X)$ is a scheme quasi-compact separated (not finite type) / $E(G,X)$.

There are two approaches to Shimura varieties:

(1) $\mathbb{C}$-mnfld $\rightsquigarrow$ $\mathbb{C}$-variety $\xrightarrow{\text{Weil descent}}$ Sh. var$/E(G,X)$. (lecture 2)

(2) moduli problem $\underset{\text{representable}}{\overset{\text{show}}{\rightsquigarrow}}$ $Sh/E(G,X)$ (lecture 2)

3. Shimura varieties over $\mathbb{C}$:

Define, as a set,

$$Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty) / K$$

where $\gamma(x,g)k = (\gamma x \gamma^{-1}, \gamma g k)$

Define $g': Sh_{K'}(G,X) \longrightarrow Sh_K(G,X)$ $\quad g \in G(\mathbb{A}^\infty)$ iff

$$(x,g) \longmapsto (x, gg').$$

This is well-defined if $(g')^{-1}K'g' \subseteq K$.

Borel: $G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty)/K$ finite set.

So one can identify $Sh_K(G,X)$ with finite disjoint union:

$$Sh_K(G,X) = \coprod_{i \in I} \Gamma_i \backslash X^+ \qquad X^+ = \text{connected component.}$$

Exercise: Assume $X = X^+$

$$G(\mathbb{A}^\infty) = \coprod_{i \in I} G(\mathbb{Q}) g_i K$$

$\rightsquigarrow$ can take $\Gamma_i = g_i K g_i^{-1} \cap G(\mathbb{Q})$.

__Step 1:__ There is a holomorphic structure on X^+, which induces a holomorphic structure on $\Gamma_i \backslash X^+$, and so a holo. structure on $Sh_K(G,X)$.

Baily-Borel: $\exists$ canonical open emb.

$$\Gamma_i \backslash X^+ \longrightarrow \text{a proj. } \mathbb{C}\text{-mnfld.}$$
$$(= \text{proj. } \mathbb{C}\text{-var.})$$

__Step 2:__ $Sh_K(G,X)$ is a quasi-proj. var. $/\mathbb{C}$.

__Example:__ $G = GL_2$, $X = \mathfrak{h}^{\pm}$, $K = GL_2(\hat{\mathbb{Z}})$.

$$Sh_K(G,X) = G(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}^\infty) / GL_2(\hat{\mathbb{Z}})$$

$$= GL_2(\mathbb{Q})^+ \backslash \mathfrak{h} \times \underbrace{GL_2(\mathbb{A}^\infty)} / GL_2(\hat{\mathbb{Z}})$$

($GL_2(\mathbb{Q})^+ GL_2(\hat{\mathbb{Z}})$ by strong approx.

$$\simeq \underbrace{GL_2(\mathbb{Q})^+ \cap GL_2(\hat{\mathbb{Z}})}_{SL_2(\mathbb{Z})} \backslash \mathfrak{h}$$

(similar when $G = GSp_{2n}$, $X = \mathfrak{h}_n^{\pm}, \ldots$)

__Example:__ Same G, X, but $K = K(N) = \ker\left(GL_2(\hat{\mathbb{Z}}) \to GL_2(\hat{\mathbb{Z}}/N\hat{\mathbb{Z}})\right)$

$$Sh_K(G,X) = \coprod_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N) \backslash \mathfrak{h}.$$

4. Moduli interpretation of $\mathbb{C}$-points:

Interlude: Abelian varieties/$\mathbb{C}$ (a.v.)

A = a.v./$\mathbb{C}$, dim $= n$.

$$T_0 A = \mathrm{Lie}\, A \simeq \mathbb{C}^n \xrightarrow{\exp} A$$

$\uparrow$ tangent space at id.

$\ker = \Lambda \subseteq \mathbb{C}^n$ rank $2n$ free $\mathbb{Z}$-lattice

$\Rightarrow A \simeq \mathbb{C}^n/\Lambda$.

$$\{A = \text{A.v. dim } n\} \longleftrightarrow \left\{\begin{array}{l} \Lambda \subseteq \mathbb{C}^n \text{ s.t. } \exists \\ E: \Lambda\times\Lambda \to \mathbb{Z} \quad \text{"Riemann form"} \\ \bullet\ E \text{ nondeg. alt.} \\ \bullet\ H(u,v) = E(u,iv) + iE(u,v) \text{ is} \\ \quad \text{pos. def. Herm. form } \mathbb{C}^n\times\mathbb{C}^n \to \mathbb{C} \end{array}\right\}$$

We get $\{(A,\lambda)\} \longleftrightarrow \{(\Lambda, E)\}$

$\lambda: A \to A^\vee$ polarization

Example: $SL_2(\mathbb{Z})\backslash \mathfrak{H} \longleftrightarrow \{\text{elliptic curves}/\mathbb{C}\}/\simeq$

$$\tau \longmapsto (E_\tau := \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z})$$

Example: $\coprod_{i\in(\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathfrak{H} \longleftrightarrow \{(E,\alpha) \mid \alpha: (\mathbb{Z}/N\mathbb{Z})^2 \simeq E[N]\}/\simeq$

$$(i,\tau) \longmapsto \left(E_\tau, \alpha(c,d) = \frac{ci+d\tau}{N}\right)$$

Example:

$$Sp_{2n}(\mathbb{Z})\backslash \mathfrak{H}_n \xleftrightarrow{1-1} \left\{\begin{array}{l}(A,\lambda)\\ A = \text{a.v. dim } n\\ \lambda: A \to A^\vee \text{ principal}\\ \quad\text{polarization}\end{array}\right\}/\simeq$$

$G = GSp_{2n}$

$X = \mathfrak{H}_n^{\pm}$

$K = GSp_{2n}(\hat{\mathbb{Z}})$

$$\underset{\underset{\text{Mat}_n(\mathbb{C})}{\cap}}{T} \longmapsto (\Lambda_T, E_T) \xleftrightarrow{\text{dictionary}} (A_T, \lambda_T).$$

$\Lambda_T = \mathbb{Z}^n + T\mathbb{Z}^n$

$E_T = \mathrm{Im}\, H_T$

$H_T(u,v) = u\, \mathrm{Im}(T)^{-1}\bar{v}$

Exercise:

$$\coprod_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma^{(n)}(N)\backslash \mathfrak{H}_n \xleftrightarrow{1-1} \left\{\begin{array}{l}(A,\lambda,\alpha)\\ \alpha: (\mathbb{Z}/N\mathbb{Z})^{2n} \simeq A[N]\end{array}\right\}$$

pairing from λ

write down α.

fix standard symp. pairing

symplectic similitude isom. (pairing carried over up to scalar)

5. Canonical models over $E(G,X)$:

Read: Deligne or Milne.

Basic idea:

- Consider $\begin{pmatrix}\text{special}\\ \text{points}\end{pmatrix} \subset Sh_K(\mathbb{C})$

Zar. dense

Morally these special points correspond to CM a.v.

- Require $\underbrace{\text{special points}}_{\text{prescribe Galois action}}$ defined $/\bar{\mathbb{Q}}$

Thm: (Shimura, Deligne, Milne, ...) There exist a unique canonical model for $Sh_K(G,X)/E(G,X)$. (Canonical up to unique isomorphism).

6. Motivation:

$$Sh(G,X) = \varprojlim_{K} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}^\infty)/K \quad \circlearrowleft G(\mathbb{A}^\infty)$$

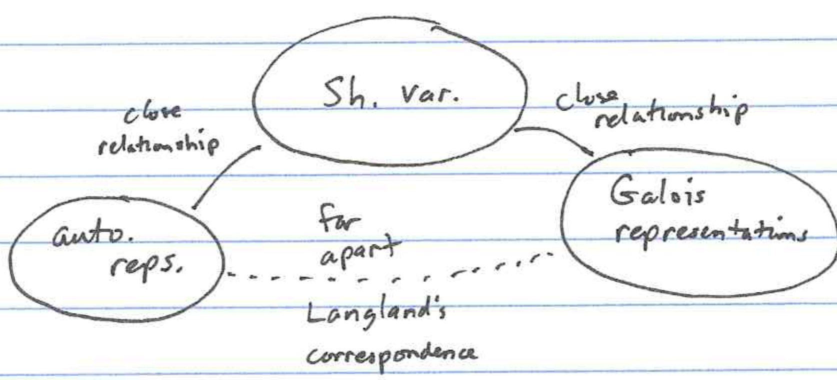
Compare with theory of automorphic forms:

$$\text{irred reps} \subseteq L^2\left(G(\mathbb{Q})\backslash G(\mathbb{A})/A_{G,\infty}\right) \circlearrowleft G(\mathbb{A}) \text{ right trans.}$$

copies of $\mathbb{R}^\times_{>0}$ in $Z(G(\mathbb{R}))$.

$\updownarrow$

auto. reps of $G(\mathbb{A})$.



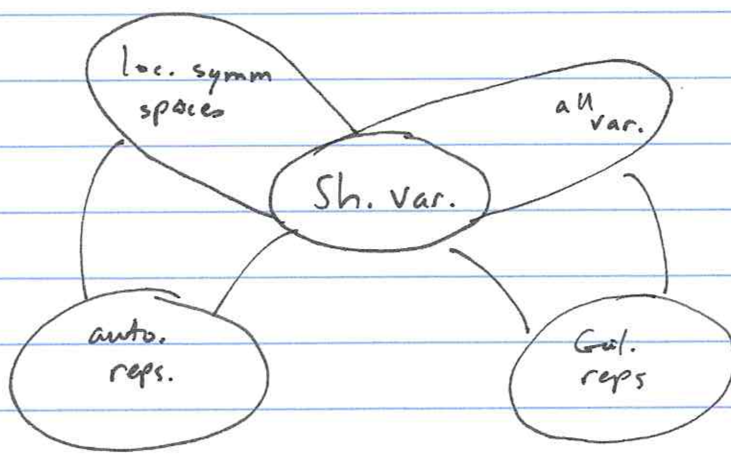
$$H_{\acute{e}t}(Sh, \mathbb{C}) = \varinjlim_{K} H_{\acute{e}t}(Sh_K \times_E \bar{E}, \mathbb{C})$$

$\circlearrowleft$

$Gal(\bar{E}/E) \times G(\mathbb{A}^{\infty})$ $= \bigoplus \begin{pmatrix} \text{aut} \\ \text{rep} \end{pmatrix} \otimes \begin{pmatrix} \text{Gal} \\ \text{rep} \end{pmatrix}$

related to Langlands.

Better handle on auto reps, can test which show up in cohom.. then attach Gal. rep. Don't know a priori which Galois reps occur.



But when you extend can't connect anymore.

PEL Shimura Varieties

Ref: Kottwitz JAMS 1992
Milne

1. What is a PEL Shimura var?

Sh. var. is of PEL type if its a moduli space of (A, λ, i, α)

A abel. scheme
λ polar. P
i endomorphism E
α level structure L

Example: $K = K(N) \subset GSp_{2n}(\hat{\mathbb{Z}})$

$$Sh_K(GSp_{2n}, \mathfrak{H}_n^{\pm}) = \coprod_{i \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \Gamma(N)\backslash \mathfrak{H}_n$$

$$\xleftrightarrow{1-1} \{(A, \lambda, \alpha)\}/\simeq$$

A: A.v. $/\mathbb{C}$ dim n; λ: pol.; α: $(\mathbb{Z}/N\mathbb{Z})^{2n} \simeq A[N]$ symp. simil.

$$\left\{\begin{array}{c}(A, \lambda, \alpha) \\ \alpha: (\mathbb{Z}/N\mathbb{Z})^{2n} \simeq A[N] \\ \text{std} \longleftrightarrow \lambda\text{-Weil}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}(A, \lambda, \alpha K) \\ \alpha: \hat{\mathbb{Z}}^{2n} \simeq TA\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}(A, \lambda, \alpha K) \\ (\mathbb{A}^{\infty})^{2n} \simeq VA\end{array}\right\}$$

up to $\mathbb{Q}^{\times}$ (on λ)

up to isom
$\simeq$ symp. simil.

$$TA = \varprojlim_n A[n](\mathbb{C}) = \prod_{\ell} T_{\ell}(A)$$

$$VA = TA \otimes_{\mathbb{Z}} \mathbb{Q}$$

up to isogeny

2. Overview:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{PEL datum}\\ (B,*,V,\langle,\rangle,h)\end{array}} \arrow[r] \arrow[dr] & \boxed{\begin{array}{c}\text{Sh. datum}\\ (G,X)\end{array}} \arrow[r, "\text{last time}"] & \boxed{\begin{array}{c}\text{Sh var.}\\ Sh_K(G,X)\\ \text{over } E(G,X)\end{array}} \arrow[d, leftrightarrow, "?"] \\
& \boxed{\begin{array}{c}\text{moduli problem}\\ \text{for } (A,\lambda,i,\alpha K)\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{moduli var.}\\ M_K(G,X)/_{E(G,X)}\end{array}}
\end{tikzcd}$$

About ?: $M_K(G,X)$ is $\#\ker^1(\mathbb{Q},G)$ - copies of $Sh_K(G,X)$. ($\ker^1$ finite)

3. PEL datum:

B = fin. dim. s.s. alg. $/\mathbb{Q}$.

$*$ = pos. involution on B

$\quad (\mathrm{tr}_{B_{\mathbb{R}}/\mathbb{R}}(bb^*) > 0 \quad \forall\, b \in B_{\mathbb{R}},\ b \neq 0)$

V = fin. free B-module

$\langle\,,\rangle : V \times V \to \mathbb{Q}$, non-deg. alt. $(B,*)$ - Hermitian

$$\langle bv_1, v_2\rangle = \langle v_1, b^* v_2\rangle \quad \forall\, b \in B,\ v_1, v_2 \in V.$$

$h : \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ $\mathbb{R}$-alg. homom. $(x,y) \mapsto \langle x, h(i)y\rangle$ is pos. def.

$$\langle h(z)v_1, v_2\rangle = \langle v_1, h(\bar{z})v_2\rangle.$$

Motivation: A = ab. var. $/\mathbb{C}$.

$\lambda : A \to A^\vee$ pol.

$i : B \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, B as above

Other datum now comes from this info. in this case.

$*$ = restriction of λ-Rosati involution.

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$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "f"'] & A^\vee \\ A \arrow[r, "\lambda"'] & A^\vee \arrow[u, "f^\vee"'] \end{tikzcd}$$

$$f \longmapsto \lambda^{-1} f^\vee \lambda$$
λ-Rosati

$V = H_1(A, \mathbb{Q})$.

$\langle , \rangle$: Riemann form

$\mathbb{C}$ - v.s. of dim n.

$h: \mathbb{C} \to H_1(A, \mathbb{R}) \simeq \mathrm{Lie}(A)$

so h is just giving complex structure on $\mathrm{Lie}(A)$.

4. PEL $\to$ Sh. datum:

$\forall$ R : $\mathbb{Q}$-alg.

$G(R) = \{ (g, \lambda) \in \mathrm{End}_{B_R}(V_R) \times R^\times : \langle gv_1, gv_2 \rangle = \lambda \langle v_1, v_2 \rangle$
$\forall v_1, v_2 \in V_R \}$

simil. group for $(V, \langle , \rangle)$.

$\underline{X}$: $\quad h: \mathbb{C} \to \mathrm{End}_{B_\mathbb{R}}(V_\mathbb{R})$

$\rightsquigarrow h: \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_\mathbb{R}$

$X = G(\mathbb{R})$ - conjugacy classes of h.

This (G, X) satisfies the axioms for Shimura datum.
(see Kottwitz article Lemma 4.1).

$h \rightsquigarrow h_\mathbb{C} : \mathbb{G}_m \times \mathbb{G}_m \to G_\mathbb{C}$
$\quad z \quad \bar{z}$

$\mu : \mathbb{G}_m \longrightarrow G_{\mathbb{C}}$ (up to $G(\mathbb{C})$-conj.)

$E(G,X)$ = field of def. of $G(\mathbb{C})$-conj. class of μ.

($H = \{\sigma \in Aut(\mathbb{C}/\mathbb{Q}) : \mu^\sigma$ and μ are conj.$\}$,

$E(G,X)$ = (missed this ☹)

<u>Example:</u> (symplectic)

$B = \mathbb{Q}$ $* = 1$ $V = \mathbb{Q}^{2n}$, $\langle , \rangle$ standard symp. pairing

$h : \mathbb{C} \longrightarrow End_{\mathbb{R}}(\mathbb{R}^{2n}) = Mat_{2n}(\mathbb{R})$

$z = a+bi \longmapsto \begin{pmatrix} aI_n & -bI_n \\ bI_n & aI_n \end{pmatrix}$

$\mu : z \mapsto \begin{pmatrix} zI_n & 0 \\ 0 & I_n \end{pmatrix}$.

Then $G = GSp_{2n}/\mathbb{Q}$.

$X = \mathfrak{h}_n^{\pm}$ dim $\frac{n(n+1)}{2}$

$E(G,X) = \mathbb{Q}$.

<u>Example:</u> (unitary)

$B = E$ (imag. quad. field)

$* = c$ = complex conj.

$V = E^n$

$\langle , \rangle : E^n \times E^n \longrightarrow \mathbb{Q}$ $p+q=n$.

$(v_1, v_2) \longmapsto \underset{tr_{E/\mathbb{Q}}}{\wedge}\left(v_1 \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}^t v_2^c\right)$

$h : \mathbb{C} \rightarrow End_{B_{\mathbb{R}}}(V_{\mathbb{R}}) \simeq Mat_n(\mathbb{C})$. : $z \longmapsto \begin{pmatrix} zI_p & \\ & \bar{z}I_q \end{pmatrix}$.

$\mu : z \longmapsto \begin{pmatrix} zI_p & \\ & I_q \end{pmatrix}$

$G = GU(p,q)$

X $\dim = pq$

$$E(G,X) = \begin{cases} E & p \neq q \\ \mathbb{Q} & p = q \end{cases}$$

5. Classification of PEL simple data:

Assume B = simple $\mathbb{Q}$-alg.

$\rightsquigarrow$ $Z(B)$ = field 2^* pos. inv.

$\rightsquigarrow$ $Z(B)$ = tot. real $* = 1$ 1^{st} kind

or

CM $* = c$ 2^{nd} kind

type	$*$	$\mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$	G
A	2^{nd}	$\mathrm{Mat}_n(\mathbb{C})$	unitary
C	1^{st}	$\mathrm{Mat}_{2n}(\mathbb{R})$	symplectic
D	1^{st}	$\mathrm{Mat}_n(\mathbb{H})$	orthogonal

↑ Hamitonian

6. PEL Moduli problem:

S = scheme $\quad (Sch/_S) \xrightarrow[\text{fully faithful}]{\text{Yoneda}} \mathrm{Functor}(Sch/_S, Sets)$

$$X \longmapsto (S' \overset{h_X}{\longmapsto} \mathrm{Hom}(S', X)).$$

Moduli problem: a functor $(Sch/_S) \to Sets$

$\mathcal{F}$ $\quad S' \longmapsto \left(\text{geom. obj's}/S'\right)$.

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Say $\mathcal{F}$ is representable by a scheme T over S if $\mathcal{F} \simeq h_T$.
($\mathcal{F}$ "is" a scheme.)

PEL moduli problem for $(B, *, V, \langle , \rangle, h)$ (at ∞ level)

$$M \ : \ (Sch/_{E(G,X)}) \longrightarrow Sets$$

$$(M := \varprojlim M_K) \qquad S \longmapsto \{(A, \lambda, i, \alpha)\}/\text{isog} \sim$$

A = ab. sch./S

$\lambda : A \to A^\vee$ $\mathbb{Q}^\times$-orbit of polar. (in $\mathrm{Hom}_S(A, A^\vee) \otimes_{\mathbb{Z}} \mathbb{Q}$).

$i : (B, *) \longrightarrow (\mathrm{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}, \lambda\text{-Rosati})$

$\mathbb{Q}$-alg. mor. w/ invol.

$$\alpha : (V \otimes_{\mathbb{Q}} \mathbb{A}^\infty, \langle , \rangle)_S \underset{\uparrow}{\simeq} (VA, \lambda\text{-Weil})$$

symp simil. isom.
compatible w/ $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-mod. action.

(condition on Hodge structure) $\exists$ $B \otimes_{\mathbb{Q}} \mathcal{O}_S$-mod isom

$V_1 \otimes_{\mathbb{Q}} \mathcal{O}_S \simeq \mathrm{Lie}(A)$ $\circlearrowleft$ $B \otimes \mathcal{O}_S$.

$$V_{\mathbb{C}} = V_1 \oplus V_2$$

$h(z)$ $\circlearrowleft$ $V_{\mathbb{C}}$; $h(z)$ acts by z on V_1; acts by $\bar{z}$ on V_2

Remark: A = a.v. /$\mathbb{C}$ $\quad V = H_1(A, \mathbb{Q})$

$$V_{\mathbb{C}} \simeq \underset{V_1}{\mathrm{Lie}(A)} \oplus \underset{V_2}{\overline{\mathrm{Lie}(A)}}$$

Def of isogeny:

$(A, \lambda, i, \alpha) \sim (A', \lambda', i', \alpha') \iff \exists\ f : A \to A'$ isog. carrying (λ, i, α) to (λ', i', α').

Remark: $G = GSp_{2n}$, $B = \mathbb{Q}$

i does nothing $\rightsquigarrow$ can get rid of it.

$G(\mathbb{A}^\infty)$ - action on M

$\forall\, g \in G(\mathbb{A}^\infty)$, $(A, \lambda, i, \alpha) \overset{g}{\longmapsto} (A, \lambda, i, \alpha \circ g)$.

$\rightsquigarrow$ Hecke correspondences

To define M_K, in the moduli problem replace α by the $\pi_1(S)$-inv. K-orbit αK

Thm: Suppose K is sufficiently small, $K \subseteq G(\mathbb{A}^\infty)$ (K is "neat"), then M_K is represented by a quasi-projective smooth variety over $E(G, X)$.

References: [Chai - Faltings] Chap 1.4 } Siegel case.
[Geometric Invariant Theory] Chap. 7
[Kottwitz] §5 (Siegel $\Rightarrow$ PEL)
[Lan Thesis] Chap 2 (any PEL)
[Shimura] Annals paper

Further topics:

- integral models
  - good red.
  - bad red.
- compactifications
  - $\mathbb{C}$
  - $E(G, X)$
  - integral
- p-adic geometry

Cohomology revisited:

Recall $H_{et}(Sh) = \bigoplus \pi \otimes R(\pi)$

$$\underset{G(\mathbb{A}^\infty)}{\circlearrowleft} \qquad \underset{Gal(\bar{E}/E)}{\circlearrowleft} \qquad E = \text{reflex field.}$$

Related to Langlands.

Hypothesis: π is "stable - tempered"
(no endoscopy)

$$h \rightsquigarrow \mu : \mathbb{G}_m \longrightarrow G_{\mathbb{C}}.$$

$$\rightsquigarrow (r_\mu, V_\mu) \qquad r_\mu : {}^L G \longrightarrow GL(V_\mu) \quad \mathbb{C}\text{-v.s.}$$

$$r_\mu|_{\hat{G}} = \text{highest weight } \mu.$$

Let $v | p$. place of E.

$$\pi_p \rightsquigarrow W'_{\mathbb{Q}_p} \longrightarrow {}^L G_{\mathbb{Q}_p}$$

Local Langlands corresp.

$$\xrightarrow{\text{res.}} W'_{E_v} \longrightarrow {}^L G_{E_v} \xrightarrow{r_\mu} GL(V_\mu)$$

$$\varphi_{\pi_p}$$

Recipe

$$WD(R(\pi)|_{Gal(\bar{E}_v/E_v)}) = \varphi_{\pi_p}.$$

For $GU(p,q)$ get $\binom{n}{p}$ dim Galois reps.

For GSp_{2n} get 2^n dim Galois reps.