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An overview of the theory of Eisenstein series:

1. Intertwining operators
2. Definitions & state the theorem (meromorphy, functional equation, holomorphy on imaginary axis)
3. Constant terms
4. Proof of meromorphic cont. (Moeglin - Waldspurger)
5. Proof of functional equation
6. Absolute convergence.

Intertwining Operators:

(Reference chap 4 book)

k = local field, include arch. places.
G = connected reductive group / k.
If k = p-adic, $\mathcal{O}_k$ = ring of integers, $q_k = |\mathcal{O}_k / \mathfrak{p}_k|$,
$|\varpi_k| = q_k^{-1}$.

fixed minimal parabolic

$P_0 = M_0 N_0$ $\qquad$ $P \supset P_0 \iff$ standard.
$P_0 = B$ = Borel subgroup if quasi-split.

A_0 = split component (of the center) of M_0.
Φ^+ = positive roots of A_0 in N_0.
Δ = simple roots.

$N_P = N$
$P/N \simeq M \supset M_0$

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roots of A_0 in M $< \Sigma(A_0, G)$
$\theta \subset \Delta$, M generated by simple roots in θ
$M = M_\theta$ $\quad W_0 = W(A_0, G)$ Weyl group
$W(\theta, \theta') = \{ \tilde{w} \in W_0 : \tilde{w}(\theta) = \theta' \}$
We say θ, θ' are associate iff $W(\theta, \theta') \neq \phi$.
Let w represent $\tilde{w}$.
$N_w = N_0 \cap w N_\theta^- w^{-1}$, N_θ^- = opposite of $N_\theta = N = N_P$.

$N_w \simeq {}_{wNw^{-1} \cap N'} \backslash N'$ (also correct for k-rational points)

$(\sigma, \mathcal{H}(\sigma))$ = irreducible admissible rep. of $M(k)$.
Assume σ is Banach.
k = arch, then $(\sigma_\infty, \mathcal{H}(\sigma)_\infty)$. differentiable representation of $M(k)$ on differentiable vectors.

$\mathfrak{a}_M = \mathfrak{a}_\theta = \mathrm{Hom}(X(M)_k, \mathbb{R})$ = real Lie ~~alg~~ algebra of A.
A = split component of M. $= A_M$.
This allows one to define a log:

$$H_M : M(k) \longrightarrow \mathfrak{a}_\theta$$

$$\exp \langle H_M(m), \underset{\underset{X(M)_k}{\cap}}{\chi} \rangle = |\chi(m)|$$

$\mathfrak{a}_{M,\mathbb{C}} = \mathfrak{a}_M \otimes_{\mathbb{R}} \mathbb{C}$ complexification

$\mathfrak{a}_M^* = X(M)_k \otimes_{\mathbb{Z}} \mathbb{R}$ dual

$\nu \in \mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_M \otimes_{\mathbb{R}} \mathbb{C}$.

$I(\nu, \sigma) = \mathrm{Ind}_{M(k)N(k)}^{G(k)} \sigma \otimes \exp\langle \nu, H_M(\cdot)\rangle \otimes \mathbb{1}$

$f \in V(\nu, \sigma)$ = space of $I(\nu, \sigma)$

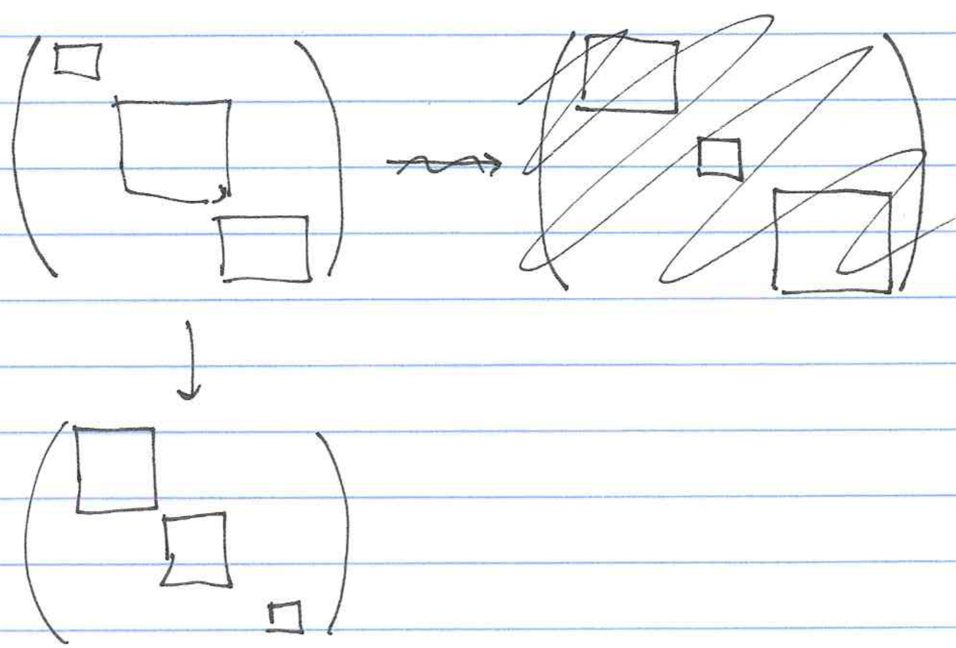
$$A(\nu, \sigma, w) f(g) = \int_{N_w(k)} f(w^{-1} n g)\, dn$$

$A(\nu, \sigma, w) : I(\nu, \sigma) \longrightarrow I(w(\nu), w(\sigma))$ Intertwining operator.

This converges absolutely ~~for~~ for $\mathrm{Re}\langle \nu, H_\alpha \rangle \gg 0$
for $\alpha \in \Delta$ s.t. $w(\alpha) < 0$.

When σ is tempered, then convergence is for $\mathrm{Re}\langle \nu, H_\alpha \rangle > 0$.

We want to reduce this to rank one situation.



Lemma: $\Theta, \Theta' \subset \Delta$, $\tilde{w} \in W(\Theta, \Theta') \neq \emptyset$. Then there exist $\Theta_1, \dots, \Theta_n \subset \Delta$ s.t.

a) $\Theta_1 = \Theta$, $\Theta_n = \Theta'$

b) fix $1 \le i \le n-1$, $\exists\, \alpha_i \in \Delta - \Theta_i$ s.t. Θ_{i+1} is conjugate to Θ_i in $\Omega_i = \Theta_i \cup \{\alpha_i\}$.

c) Set $\tilde{w}_i = \tilde{w}_{k,\Omega_i} \quad \tilde{w}_{k,\Theta_i} = (\tilde{w}_{k,\Theta_i})^{-1}$
$$\in W(\Theta_i, \Theta_{i+1})$$
for $1 \le i \le n-1$, then $\tilde{w} = \tilde{w}_{n-1} \cdots \tilde{w}_1$.

(depends on the parabolic)

d) Set $\tilde{w}_1' = \tilde{w}_1$ and $\tilde{w}_{i+1}' = \tilde{w}_i' \cdot \tilde{w}_i^{-1}$ $\quad 1 \le i \le n-1$.

then $\tilde{w}_n' = 1$ and $\bar{n}_{w_i'} = \bar{n}_{w_i} \oplus Ad(w_i^{-1})\bar{n}_{w_{i+1}'}$

$\bar{N}_w = w^{-1} N_w w \quad \bar{n}_w = Lie(\bar{N}_w)$.

Theorem: $\Theta, \Theta' \subset \Delta$, $\tilde{w} \in W(\Theta, \Theta')$., $\Theta_1, \dots, \Theta_n \subset \Delta$, $\tilde{w}_i \in W(\Theta_i, \Theta_{i+1})$. Assume ν is inside an appropriate cone of positive Weyl chamber. so that $A(\nu, \sigma, w)$ is absolutely convergent. $\nu_i = \tilde{w}_{i-1}(\nu_{i-1})$, $\nu_1 = \nu$. $2 \le i \le n-1$, $\sigma_i = w_{i-1}(\sigma_{i-1})$, $\sigma_1 = \sigma$, $2 \le i \le n-1$.

Then
$$A(\nu, \sigma, w) = A(\nu_{n-1}, \sigma_{n-1}, w_{n-1}) \cdots A(\nu_1, \sigma_1, w_1).$$

Example: (Unramified case)

$G =$ split reductive group. , $k = p$-adic

$B = TU =$ Borel $= P_0$.

$\Theta = \Theta' = \phi$.

$A_0 = T$

$\tilde{w} = \tilde{w}_k =$ long element $\in W(T, G)$

$\sigma = \mu =$ unramified character of T; i.e., $\mu|_{T(\mathcal{O}_k)} = 1$.

$N = U = N_{w_\ell}$

$f_\circ \in V(\mu)$, $f_\circ(k) = f_\circ(e) = 1$.

$$A(o, \mu, w_\ell) f_\circ(e) = \int f_\circ(w_\ell^{-1} u w_\ell)\, du = \int_{\bar{U}} f_\circ(\bar{u})\, d\bar{u}.$$

$\alpha > 0$ a positive root.

We have a map

$$\xi_\alpha : SL_2 \longrightarrow G$$

$$\langle \xi_\alpha\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right), \xi_\alpha\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right), \xi_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \rangle = G_\alpha.$$

$$\xi_\alpha\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) = \langle \exp(tX_\alpha) \rangle = U_\alpha, \qquad \xi_\alpha\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) = \exp(tX_{-\alpha})$$

$\tilde{w}_\ell = \tilde{w}_{\alpha_n} \cdots \tilde{w}_1$, $\tilde{w}_i =$ regular simple reflection.

So it reduces to computing an integral of the form

$$\int_k f\left(\xi_\alpha\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) dx = \text{single rank one operator}$$

normalize as $\text{Meas}\left(\xi_\alpha\begin{pmatrix} 1 & 0 \\ \mathcal{O}_k & 1 \end{pmatrix}\right) = 1$.

$$\int_k f\left(\xi_\alpha\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) dx = \frac{1 - q^{-1}\mu(a_\alpha)}{1 - \mu(a_\alpha)}.$$

$a_\alpha =$ coset of $\begin{pmatrix} \varpi_k & 0 \\ 0 & \varpi_k^{-1} \end{pmatrix}$ modulo $\begin{pmatrix} \mathcal{O}_k^* & 0 \\ 0 & \mathcal{O}_k^* \end{pmatrix}$

$$= \underbrace{\int_{x \in \mathcal{O}_k} du}_{=1} + \int_{x \notin \mathcal{O}_k} f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) dx \qquad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x^{-1} \end{pmatrix}$$

$$\int_{x \notin \mathcal{O}} f\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right) dx = \int_{x \notin \mathcal{O}} f\left(\begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix}\right) dx$$

$$= \int_{x \notin \mathcal{O}} \mu(x^{-1})\, |x|^{-1}\, dx$$

(get sum over
$(p^{-1} - \mathcal{O}) \cup (p^{-2} - p^{-1}) \cup \cdots$)

So the ~~[illegible]~~ we get becomes

$$1 + \sum_{n \geq 1} \mu(\varpi^{n})\, q^{-n} \underbrace{\int_{(\varpi^{-n}) - (\varpi^{-n+1})} |x|\, dx}_{q^{n}\left(\int_{\mathcal{O}^{\times}} dx^{\times}\right) = (1 - q^{-1})}.$$

$$A(\mu, w_\ell)\, f_0(e) = \prod_{\alpha \in \Sigma^{+} - \Sigma_M^{+}} \left(\frac{1 - q^{-1}\mu(a_\alpha)}{1 - \mu(a_\alpha)} \right).$$

r = adjoint action of ${}^{L}T$ on ~~[illegible]~~ ${}^{L}\mathfrak{n} = \mathrm{Lie}({}^{L}N)$ $N = U$.

$L(s, \mu, \tilde{r})$ = Langland's L-fctn attached to $\tilde{r}$ = contragredient of r.

$$A(\mu, w_\ell)\, f_0(e) = \frac{L(0, \mu, \tilde{r})}{L(1, \mu, \tilde{r})}.$$

$r = \bigoplus_{i=1}^{m} r_i$ irreducible decomposition

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$\tilde{\alpha} = \langle \tilde{\rho}_P, \alpha \rangle^{-1} \rho_P$

$$\underset{\substack{\shortparallel \\ \nu}}{A(s\tilde{\alpha}}, \underset{\substack{\shortparallel \\ \mu}}{\sigma}, \underset{\substack{\wedge \\ f_0(e)}}{w_\ell)} = \prod_{i=1}^{m} \left(L(is, \sigma, \tilde{r}_i) / L(1+is, \sigma, \tilde{r}_i) \right).$$

Same set-up as last time:

G, $P_0 = M_0 N_0$ minimal parabolic, G defined over $\mathbb{Q}$.

$P = MN$ $\quad M \supset M_0,\ N \subset N_0$.

$\mathfrak{a}_G$ $\quad H_G : G(\mathbb{A}) \longrightarrow \mathfrak{a}_G$.

$G(\mathbb{A})^1 = \ker(H_G)$.

$G(\mathbb{A}) = G(\mathbb{A})^1 A_G(\mathbb{R})^0$

π = irreducible unitary rep. of $G(\mathbb{A})$.

$\nu \in \mathfrak{a}^*_{G,\mathbb{C}}$

$\pi_\nu(x) = \pi(x) \exp\langle \nu, H_G(x)\rangle$. $\leftarrow$ π twisted by ν.

$R_{M,disc}$ = right regular action of $M(\mathbb{A})$ on $L^2_{disc}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$.

$y \in G(\mathbb{A})$

$I_P(\nu) = R_{M,disc} \exp\langle \nu, H_M(\cdot)\rangle$

V_P = space of rep. consists of all measurable fctns ϕ s.t.

$$\phi : N(\mathbb{A})M(\mathbb{Q})A_M(\mathbb{R})^0 \backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

$m \in M(\mathbb{Q})\backslash M(\mathbb{A})^1 \qquad \phi_x : m \longmapsto \phi(mx)$

to belong to $L^2_{disc}(M(\mathbb{Q})\backslash M(\mathbb{A})^1)$ for almost all x.

$G(\mathbb{A}) = P(\mathbb{A})K$ $\leftarrow$ max. cmpct

and

$$\|\phi\|^2 = \int_K \int_{M(\mathbb{Q})\backslash M(\mathbb{A})^1} |\phi(mk)|^2 \, dm\, dk < \infty.$$

Note the space defined here does not have any reference to ν. The ν will now show up in the action.

$$(I_P(\nu,y)\phi)(x) = \phi(xy) \exp\langle \nu+\rho_P, H_P(xy)\rangle \exp\langle -(\nu+\rho_P), H_P(x)\rangle.$$

If $P, P' \supseteq P_0$ are parabolics, then $\mathfrak{a}_P = \mathfrak{a}_M$, $\mathfrak{a}_{P'} = \mathfrak{a}_{M'}$

$W(\mathfrak{a}_P, \mathfrak{a}_{P'}) = W(\theta, \theta')$ for $M = M_\theta$, $M' = M_{\theta'}$.

$$\rho_P = \frac{1}{2} \sum_{X_\alpha \in \mathrm{Lie}(N)} \alpha.$$

Def: $x \in G(\mathbb{A})$, $\phi \in V_P$, $\nu \in \mathfrak{a}^*_{M,\mathbb{C}}$

$$E(x, \phi, \nu) = \sum_{\delta \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \phi(\delta x) \exp\langle \nu + \rho_P, H_P(\delta x)\rangle$$

is an Eisenstein series attached to ϕ. $\underbrace{\phi(\ ) \exp\langle \nu + \rho_P, H_P(\ )\rangle}_{=:\phi_\nu}$

$s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ w_s = rep. in $G(\mathbb{Q})$.

$$M(s,\nu) : V_P \to V_{P'}$$

$$(M(s,\nu)\phi)(x) = \int_{N'(\mathbb{A}) \cap w_s N(\mathbb{A}) w_s^{-1} \backslash N'(\mathbb{A})} \phi(w_s^{-1} n x) \exp\langle \nu + \rho_P, H_P(w_s^{-1} n x)\rangle \cdot \exp\langle -(s\nu + \rho_{P'}), H_{P'}(x)\rangle \, dn$$

This $M(s,\nu)$ is an intertwining between $I_P(\nu)$ and $I_{P'}(s\nu)$.

Then

$$E(x, I_P(\nu, y)\phi, \nu) = E(xy, \phi, \nu).$$

For this definition of E.S. to be useful, one needs to be careful with the choice of ϕ.

Take fctns in V_P which are K-finite, $\mathfrak{Z}_\infty$-finite

($\mathfrak{Z}_\infty$ = center of $\mathcal{U}(\mathrm{Lie}(G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C})$)

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Denote this set of fctns by V_P°. V_P° is dense in V_P.

$(\mathfrak{a}_P^*)^+ = \{ \nu \in \mathfrak{a}_P^* \mid \langle \nu, \alpha^\vee \rangle > 0 \}$.
$\forall \alpha$ s.t. $X_\alpha \in Lie(N)$

Convergence for all $\nu \in \mathfrak{a}_{P,\mathbb{C}}^*$ $Re(\nu) \in (\mathfrak{a}_P^*)^+ + \rho_P$.
$\mathcal{C} = \{$such $\nu\}$.

<u>Theorem : (Langlands)</u> Suppose $\phi \in V_P^\circ$. Then $E(x, \phi, \nu)$, $M(s,\nu)\phi$ can be extended to meromorphic functions on $\mathfrak{a}_{P,\mathbb{C}}^*$ satisfying

$$E(x, M(s,\nu)\phi, s\nu) = E(x, \phi, \nu)$$
$$M(ts, \nu) = M(t, s\nu) M(s,\nu) \qquad s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'}), \quad t \in W(\mathfrak{a}_{P'}, \mathfrak{a}_{P''})$$

If $\nu \in \sqrt{-1}\,\mathfrak{a}_P^*$, then both E & M are analytic & $M(s,\nu)$ extends to a unitary operator from V_P to $V_{P'}$.

P, P' $\quad P = MN$, $P' = M'N'$ $\quad P, P' \supset P_0$.

<u>Def:</u> Constant term along P' of $E(x,\phi,\nu)$ is

$$E_{P'}(x,\phi,\nu) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} E(nx, \phi, \nu)\, dn.$$

Will do the computation of constant terms when $rk\, P = rk\, P'$
P, P' max. rank $= \dim (A_? \backslash A_M)$.

$$P(\mathbb{Q}) \delta N'(\mathbb{Q}) = \coprod_{\gamma \in \delta^{-1} P(\mathbb{Q}) \delta \cap N'(\mathbb{Q}) \backslash N'(\mathbb{Q})} P(\mathbb{Q}) \delta \gamma$$

$$E_{P'}(x,\phi,\nu) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\nu(\delta n x)\, dn$$

$$= \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})} \sum_{\gamma \in \delta^{-1}P(\mathbb{Q})\delta \cap N'(\mathbb{Q})\backslash N'(\mathbb{Q})} \phi_\nu(\delta \gamma n x)\, dn$$

$$= \int_{\delta^{-1}P(\mathbb{Q})\delta \cap N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})/N'(\mathbb{Q})} \phi_\nu(\delta n x)\, dn$$

(Now use Bruhat decomp)

A_0 = split comp. of N_0.

$$\left(G(\mathbb{Q}) = \bigcup_{w \in N_G(A_0) \cap G(\mathbb{Q})} P(\mathbb{Q}) w P'(\mathbb{Q}). \right)$$

So we can write $\delta = w\gamma'$ $\gamma' \in P'(\mathbb{Q})/N'(\mathbb{Q})$.

Typical integral

$$\int_{\gamma'^{-1} w^{-1} P(\mathbb{Q}) w \gamma' \cap N'(\mathbb{Q})\backslash N'(\mathbb{A})} \phi_\nu(w\gamma' n \gamma'^{-1} \gamma' x)\, dn$$

$$= c(\gamma') \int_{w^{-1} P(\mathbb{Q}) w \cap N'(\mathbb{Q})\backslash N'(\mathbb{A})} \phi_\nu(w n \gamma' x)\, dn$$

Constant depends on measures.

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$\phi^+ = \phi^+(A_0, G) =$ pos. roots in N_0.

$\phi^+(A_0, M) \subset \phi^+$.

$w(\phi^+(A_0, G))$ will define another positive chamber. and by restriction a chamber C_M for $W(A_0, M)$.

By transitivity of the action of $W(A_0, M)$, $\exists$ w_1 s.t. $w_1(C_M)$ is the positive chamber as above.

Conclusion: positive roots of A_0 in M are of the form $w(\alpha)$, $\alpha \in \phi^+(A_0, G)$

$$\phi_\theta^1 = \{ \alpha \in \phi_\theta^+ \mid \alpha = w(\beta) \quad \beta \in \phi^+ - \phi^+_{\theta'}. \}$$

$$\phi_\theta^2 = \{ \alpha \in \phi_\theta^+ \mid \alpha = w(\beta) \quad \beta \in \phi^+_{\theta'} \}.$$

$\alpha, \alpha' \in \phi_\theta^1$, $\alpha + \alpha'$ is a root, then $\alpha + \alpha' \in \phi_\theta^1$ & if $\alpha \in \phi_\theta^1$ and $\alpha' \in \phi_\theta^2$, $\alpha + \alpha'$ is a root, $\alpha + \alpha' \in \phi_\theta^1$.

$N_1 =$ subgroup of M, unipotent, generated by X_α, $\alpha \in \phi_\theta^1$.
Then this is a unipotent radical of a parabolic in M.

$$w^{-1} N_1 w \subset N', \qquad w^{-1} P(\mathbb{Q}) w \cap w^{-1} N_1(\mathbb{Q}) w = w^{-1} N_1(\mathbb{Q}) w$$

$$\int_{n \in w^{-1}P(\mathbb{Q})w \cap N'(\mathbb{Q}) w^{-1} N_1(\mathbb{A}) w \backslash N'(\mathbb{A})} \left(\int_{n_1 \in N_1(\mathbb{Q}) \backslash N_1(\mathbb{A})} \phi(n_1 w n \gamma x') \, dn_1 \right) dn$$

Assume $\phi_y : m \longmapsto \phi(my) \in L^2_{cusp}(M(\mathbb{Q}) \backslash M(\mathbb{A})^1)$

$y = wn\gamma' x$, then the inner integral vanishes unless $N_1 = \{0\}$.

$\Rightarrow \phi'_\theta = \phi \quad \forall\, \alpha \in \Phi^+_\theta$ is a $w(\beta)$, $\beta \in \Phi^+_{\theta'}$ $\Rightarrow M' = w^{-1} M w$.

Since they are both maximal $\overset{?}{=}$ rank, $\gamma' \in P'(\mathbb{Q})/N'(\mathbb{Q}) \simeq M'(\mathbb{Q})$

$M'(\mathbb{Q}) = w^{-1} M(\mathbb{Q}) w$, $\delta = w\gamma'$, $\gamma' = 1$

(max so α = unique simple root in N)

There are two possibilities:

1) $w(\alpha) > 0$ since $w(\theta) > 0 \Rightarrow w = 1$. $\Rightarrow P = P'$. cannot happen unless $1 \in W(\sigma_P, \sigma_{P'})$

2) $w(\alpha) < 0$ then $wP'w^{-1}$ is the parabolic subgroup opposed to P. Then $w^{-1} P(\mathbb{Q}) w \cap N'(\mathbb{Q}) = \{1\}$.

$$\Rightarrow \int_{N'(\mathbb{A})} \phi_v(wnx)\,dn$$

P, P' max parabolics $\supset P_0$.

$W(\sigma_P, \sigma_{P'}) \neq \phi$.

Suppose $P = P'$. $W(\sigma_P, \sigma_{P'}) = \{\pm 1\}$. or $W(\sigma_P, \sigma_P) = \{1\}$.

If $\{\pm 1\}$, P is self-associate. $-1 = w_0 = w_\ell w_{\ell,M}^{-1}$.

Moreover, $w_0 P w_0^{-1} = P^-$.

In the 2^{nd} case $w_0 P w_0^{-1} = (P')^-$, $P' \neq P$.

Thus, in the final form of the constant, there will be one or two terms depending on P' not being self-associate or otherwise.

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<u>Theorem:</u> $P, P' \supset P_0$

$$E_{P'}(x,\phi,\nu) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} E(nx,\phi,\nu)\,dn$$

$$= \sum_{s\in W(\mathfrak{a}_P,\mathfrak{a}_{P'})} (M(s,\nu)\phi)(x)\exp\langle s\nu+\rho_{P'}, H_{P'}(x)\rangle.$$

self-associate $\quad \phi_\nu(x) + \int_{N'(\mathbb{A})} \phi_\nu(wnx)\,dn$

not self-associate $\quad (P')^{-1} = w_0 P w_0^{-1}$

$$= \int_{N'(\mathbb{A})} \phi_\nu(wnx)\,dn.$$