Cycles on unitary Shimura varieties and derivatives of L-functions:

Kudla - Rapoport - Yang: "On the derivative of a wt 1 Eisenstein series"

$K = \mathbb{Q}(\sqrt{d_K})$ quadratic imaginary

Def: $\begin{array}{c} F \\ | \\ K \end{array}$ a CM elliptic curve/F is an elliptic curve E with $\kappa_E : \mathcal{O}_K \to \mathrm{End}(E)$ s.t. $\mathcal{O}_K \longrightarrow \mathrm{End}_F(\mathrm{Lie}(E)) = F$ is the inclusion $K \hookrightarrow F$.

If $S \to \mathrm{Spec}(\mathcal{O}_K)$ is an $\mathcal{O}_K$-scheme, a CM elliptic curve E/S is an elliptic curve E/S with $\mathcal{O}_K \to \mathrm{End}(E)$ s.t. $\mathcal{O}_K \longrightarrow \mathrm{End}_S(\mathrm{Lie}(E)) \simeq S$ is the structure map.

Special endomorphisms

$$L(E) = \{ j \in \mathrm{End}(E) : j \circ x = \bar{x} \circ j \quad \forall \; x \in \mathcal{O}_K \}$$

$\mathcal{O}_K$ acts by $x \cdot j = x \circ j$. quadratic forms $\deg(j)$.

- $K \longrightarrow \mathbb{C}$ Given CM E, $\mathcal{O}_K \longrightarrow \mathrm{End}(E)$ is an isomorphism and $\mathrm{End}(E)$ commutative $\Rightarrow L(E) = 0$.

Fix $\mathfrak{p} \subset \mathcal{O}_K$, $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$. $\begin{array}{c} \mathfrak{p} \\ | \\ (p) \subseteq \mathbb{Z} \end{array}$

- Suppose E is defined over $\overline{\mathbb{F}}_{\mathfrak{p}}$ and ordinary, then $\mathcal{O}_K \to \mathrm{End}(E)$ is an isomorphism and $L(E) = 0$.

- Suppose $E/\overline{\mathbb{F}}_{\mathfrak{p}}$ with E supersingular. Then

$$H = \mathrm{End}(E)\otimes\mathbb{Q} \supset \mathrm{End}(E) \xleftrightarrow{\kappa_E} \mathcal{O}_K$$

H is quaternion alg. s.t. $\forall\, \ell \le \infty$,

$$H\otimes\mathbb{Q}_\ell = \begin{cases} M_2(\mathbb{Q}_\ell) & \text{if } \ell \notin \{p,\infty\} \\ \text{div. alg.} & \text{if } \ell \in \{p,\infty\} \end{cases}$$

Noether - Skolem $\Rightarrow$ $x \longmapsto \kappa_E(x)$ $\quad K \hookrightarrow H$

$x \longmapsto \kappa_E(\bar{x})$

are conjugate by some $j \in H^\times$, i.e., $x \circ j = j \circ \bar{x}$

$\forall\, x \in K \Rightarrow j \in L(E)_\mathbb{Q} = L(E)\otimes\mathbb{Q}$.

$\Rightarrow \dim_K L(E)_\mathbb{Q} \geq 1$.

But $K \cap L(E)_\mathbb{Q} = 0$ in $H \Rightarrow H = K \oplus L(E)_\mathbb{Q}$

$\Rightarrow \dim_K L(E)_\mathbb{Q} = 1$.

$m \in \mathbb{Z}^+$

$\mathcal{Z}_m$ = moduli of pairs (E, j) over $\mathcal{O}_K$ - schemes

$\downarrow$ $\quad$ • E is CM elliptic curve

$\mathrm{Spec}(\mathcal{O}_K)$ $\quad$ • $j \in L(E)$, $\deg(j) = m$.

$\mathcal{Z}_m(\mathbb{C}) = \phi$.

Recall $\left(\frac{a,b}{\mathbb{Q}}\right)$ is the quaternion algebra $\mathbb{Q}\oplus\mathbb{Q}i\oplus\mathbb{Q}j\oplus\mathbb{Q}k$

with $i^2 = a$, $j^2 = b$, $ij = k = -ji$.

Hilbert symbol $\ell \le \infty$

$$(a,b)_\ell = \begin{cases} 1 & \text{if } ax^2+by^2=z^2 \text{ has nontrivial soln in } \mathbb{Q}_\ell. \\ -1 & \text{o/w} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \left(\frac{a,b}{\mathbb{Q}}\right) \otimes \mathbb{Q}_\ell \simeq M_2(\mathbb{Q}_\ell) \\ -1 & \text{if } \left(\frac{a,b}{\mathbb{Q}}\right) \otimes \mathbb{Q}_\ell \text{ is div. alg.} \end{cases}$$

<u>Lemma:</u> $\mathfrak{p} \subseteq \mathcal{O}_K$ $\quad (E, j) \in Z_m(\overline{\mathbb{F}}_\mathfrak{p})$ then

$\quad\ |\qquad |$

$(p) \subseteq \mathbb{Z}$ $\qquad \mathrm{End}(E) \otimes \mathbb{Q} = \left(\frac{d_K, -m}{\mathbb{Q}}\right)$.

In particular,

$$(d_K, -m)_\ell = \begin{cases} 1 & \text{if } \ell \notin \{p, \infty\} \\ -1 & \text{if } \ell \in \{p, \infty\} \end{cases}.$$

<u>Proof:</u> In $H = \mathrm{End}(E) \otimes \mathbb{Q}$ $\quad j + j^\vee$ is fixed by main involution, so is central, i.e.,

$$x \circ (j + j^\vee) = (j + j^\vee) \circ x$$
$$= \bar{x} \circ (j + j^\vee) \qquad \forall x \in K$$

$$\Rightarrow j + j^\vee = 0 \Rightarrow j^\vee = -j.$$

$m = \deg(j) = j \circ j^\vee = -j^2$.

$K \hookrightarrow H$

$\sqrt{d_K} \mapsto i$ $\quad$ then $i^2 = d_K$

$\qquad\qquad j^2 = -m$.

and $i \circ j = j \circ \bar{i} = -j \circ i$. Thus, $H \simeq \left(\frac{d_K, -m}{\mathbb{Q}}\right)$. ■

Note, we just calculated that $\deg(j) = -j^2$.

$$\mathrm{Diff}(m) = \left\{ \begin{matrix} \ell < \infty \\ \text{prime} \end{matrix} \;\middle|\; (d_K, -m)_\ell = -1 \right\}$$

① $(d_K, -m)_\infty = (-1, -1)_\infty = -1$

Hilbert reciprocity $\Rightarrow \prod_{\ell \leq \infty} (d_K, -m)_\ell = 1$.

So $|\mathrm{Diff}(m)|$ is odd and so never empty.

② If $p \in \mathrm{Diff}(m) \Rightarrow \left(\frac{d_K, -m}{\mathbb{Q}}\right) \otimes \mathbb{Q}_p$ is a quaternion division alg.

$$\mathbb{Q}(\sqrt{d_K}) \otimes \mathbb{Q}_p = K_p$$

$K_p \leftarrow$ is a field, and p nonsplit in K.

③ $Z_m(\bar{\mathbb{F}}_p) \neq \emptyset. \Rightarrow (d_K, -m)_\ell = \begin{cases} 1 & \text{if } \ell \notin \{p, \infty\} \\ -1 & \text{if } \ell \in \{p, \infty\} \end{cases}$

$\Rightarrow \mathrm{Diff}(m) = \{p\}$.

So, if $|\mathrm{Diff}(m)| > 1 \Rightarrow Z_m = \emptyset$.

and $\mathrm{Diff}(m) = \{p\} \Rightarrow Z_m$ is supported at p.

Z_m only supported in one char, only at s.s. ell. curves. Only fin. many s.s e.curves of fixed char. So we have:

<u>Cor:</u> Z_m has dimension 0 and

$$Z_m = \coprod_{z \in Z_m} \mathrm{Spec}(\mathcal{O}_{Z_m, z})$$

$\uparrow$ Artinian local $\mathcal{O}_K$-alg.

$$\widehat{\deg}\, Z_m = \sum_{\mathfrak{p} \subseteq \mathcal{O}_K} \log|\mathbb{F}_\mathfrak{p}| \sum_{z \in Z_m(\bar{\mathbb{F}}_\mathfrak{p})} \mathrm{length}(\mathcal{O}_{Z_m, z}).$$

<u>Thm (Gross, K-R-Y):</u> $\mathrm{Diff}(m) = \{p\}$ $\quad \begin{matrix} \mathfrak{p} & \subset & \mathcal{O}_K \\ \text{inert} \mid & & \mid \\ (p) & \subset & \mathbb{Z} \end{matrix}$ (ramified case is done w/ sim. formula)

① $\forall z \in Z_m(\bar{\mathbb{F}}_p)$

$$\mathrm{length}(\mathcal{O}_{Z_m, z}) = \frac{1 + \mathrm{ord}_p(m)}{2}$$

$$\left[(d_K, -m)_p = -1 \Rightarrow -m \text{ is not a norm from } K_p \Rightarrow \mathrm{ord}_p(m) = \text{odd} \right]$$

② $$|Z_m(\bar{\mathbb{F}}_p)| = \frac{|\mathcal{O}_K^\times|}{2} \cdot 2^{\#\text{prime divisors of } d_K} \cdot \#\{ \mathfrak{a} \subseteq \mathcal{O}_K \mid N(\mathfrak{a}) = \tfrac{m}{p} \}$$

$$\Rightarrow \widehat{\deg}\, Z_m = \log(p)\, |Z_m(\bar{\mathbb{F}}_p)| \left[\mathrm{ord}_p(m) + 1 \right].$$

These should be the coefficients of the derivative of an Eisenstein series. We now construct this Eisenstein series.

$K \hookrightarrow \mathbb{C}$

$E/\mathbb{C}$ CM elliptic curve

$L(E) = 0$

But $\mathcal{O}_K$ acts on $H_1(E) = H_1(E(\mathbb{C}), \mathbb{Z})$

$\uparrow$

proj. rank 1 over $\mathcal{O}_K$.

$$L_B(E) = \{ j \in \mathrm{End}_{\mathbb{Z}} H_1(E) \mid x \circ j = j \circ \bar{x} \quad \forall x \in \mathcal{O}_K \}$$

$\deg(j) = \det(j)$.

<u>Lemma:</u> $\forall\, \ell < \infty$,

$$(L_B(E) \otimes \mathbb{Z}_\ell, \deg) \simeq (\mathcal{O}_{K,\ell}, -x\bar{x})$$

<u>Proof:</u> Fix $\mathcal{O}_{K,\ell}$-module isomorphism $H_1(E) \otimes \mathbb{Z}_\ell \simeq \mathcal{O}_{K,\ell}$

$$\mathrm{End}_{\mathbb{Z}_\ell}(H_1(E) \otimes \mathbb{Z}_\ell) \simeq \mathrm{End}_{\mathbb{Z}_\ell}(\mathcal{O}_{K,\ell})$$

$\Downarrow$

$j_0 =$ complex conjugation

$$\mathrm{End}_{\mathbb{Z}_\ell}(H_1(E)\otimes\mathbb{Z}_\ell) = \mathcal{O}_{K,\ell} \oplus \underbrace{(L_B(E)\otimes\mathbb{Z}_\ell)}_{\mathcal{O}_{K,\ell}\, j_0}.$$

Desired isomorphism $x \longmapsto x \cdot j_0$. ∎

Incoherent collection of $\mathbb{Q}_\ell$ - quadratic spaces $\ell \le \infty$.

$$\mathcal{C}_\ell = \begin{cases} (K_\ell, -x\bar{x}) & \ell < \infty \\ (K_\ell, x\bar{x}) & \ell = \infty \end{cases}$$

(Incoherent means there is no global space that locally gives these spaces. This is incoherent by looking at the Hasse invariants.)

$$\mathcal{C}_{\mathbb{A}} = \prod_{\ell \le \infty} \mathcal{C}_\ell \quad , \quad \mathcal{S}(\mathcal{C}_{\mathbb{A}}) = \bigotimes_{\ell \le \infty} \mathcal{S}(\mathcal{C}_\ell) \text{ Schwartz functions.}$$

$SL_2(\mathbb{A})$ acts on $\mathcal{S}(\mathcal{C}_{\mathbb{A}})$ via the Weil representation ω.

Construction $\mathcal{S}(\mathcal{C}_{\mathbb{A}}) \longrightarrow$ Eisenstein series on $SL_2(\mathbb{A})$

$$\phi \longmapsto E(g, s, \phi)$$

$g \in SL_2(\mathbb{A})$

$s \in \mathbb{C}$.

$\chi : \mathbb{A}^\times \longrightarrow \{\pm 1\}$ attached to $K/\mathbb{Q}$ via CFT.

$$I_s(\chi) = \left\{ f_s : SL_2(\mathbb{A}) \longrightarrow \mathbb{C} : f_s\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} g \right) = \chi(a) |a|^{s+1} f_s(g) \right\}$$

Section $f_s \in I_s(\chi)$ is <u>standard</u> if $f_s\big|_{\text{max compact}} = f_0\big|_{\text{max. compact}}$.

Iwasawa decomp. $SL_2(\mathbb{A}) = \underset{\substack{\uparrow \\ \text{upper triangular}}}{B(\mathbb{A})} \cdot \text{max compact}$

$\Rightarrow$ Every $f_0 \in I_0(\chi)$ extends uniquely to a standard section.

Given f_s, form

$$E(g,s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash SL_2(\mathbb{Q})} f_s(\gamma g)$$

Eisenstein series.

Given $\phi \in S(C_{/\mathbb{A}})$, define $f_0(g) = (\omega(g)\phi)(0)$.

$\rightsquigarrow$ standard section $\rightsquigarrow E(g,s,\phi)$.

Just need to pick appropriate ϕ now.

$\phi = \otimes \phi_\ell \in S(C_{/\mathbb{A}})$ if $\ell < \infty$, $\phi_\ell = \mathrm{char}_{\mathcal{O}_{K,\ell}}$

if $\ell = \infty$ $\phi_\ell(z) = e^{-\pi z \bar{z}}$. $\leftarrow$ to get wt 1 E.S.

This gives $E(g,s) := E(g,s,\phi)$.

de-adelize to get $E(\tau,s)$ Eisenstein series on $\mathfrak{H}$ = upper half-plane, wt 1.

$E(\tau,-s) = -E(\tau,s) \Rightarrow E(\tau,0) = 0$.

Central derivative is non-holomorphic wt 1 modular ~~form~~ function.

$$E'(\tau,0) = \sum_{m=-\infty}^{\infty} c(m,v)\, q^m \qquad \tau = u+iv, \quad q = e^{2\pi i \tau}.$$

<u>Theorem (K-R-Y):</u> If $m > 0$ then $c(m,v) = c(m)$ is indep. of v and $c(m) = \widehat{\deg}\, Z_m$.

Deformation Theory:

$$\begin{array}{ccc} & \mathfrak{p} & \subseteq & \mathcal{O}_K \\ \text{inert} & | & & | \\ & (p) & \subseteq & \mathbb{Z} \end{array}$$

$\mathrm{Diff}(m) = \{p\}$

Fix $z \in Z_m(\overline{\mathbb{F}}_{\mathfrak{p}})$
$\quad\quad\quad\quad \overset{\shortparallel}{(E, j)}$

Goal: $\mathrm{length}(\mathcal{O}_{Z_m, z}) = \dfrac{1 + \mathrm{ord}_p(m)}{2}$

Let $W = W(\overline{\mathbb{F}}_{\mathfrak{p}}) = \hat{\mathcal{O}}^{unr}_{K, \mathfrak{p}}$. $\quad W/pW = \overline{\mathbb{F}}_{\mathfrak{p}}$

length unchanged by unramified base extension, so from now on

$$Z_m = Z_m \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(W).$$

ART = category of complete local Artinian W-algebras. w/ residue field $\overline{\mathbb{F}}_{\mathfrak{p}}$

$\mathrm{Def}_E : \mathrm{ART} \to \mathrm{SET}$

$\mathrm{Def}_{(E,j)} : \mathrm{ART} \to \mathrm{SET}.$

$\mathrm{Def}_E(R) = \{$ iso classes of $\tilde{E}$ w/ CM over R and $\tilde{E}/\overline{\mathbb{F}}_{\mathfrak{p}} \cong E \}$

$\mathrm{Def}_{(E,j)}(R) = \{$ iso classes of deformations $(\tilde{E}, \tilde{j})$ of (E, j) to $R\}$

$$(\tilde{E}, \tilde{j}) \in \mathrm{Def}_{(E,j)}(R) \rightsquigarrow (\tilde{E}, \tilde{j}) \in Z_m(R)$$

$$\begin{tikzcd} \rightsquigarrow \mathrm{Spec}(R) \arrow[r, "(\tilde{E}, \tilde{j})"] \arrow[d, "\exists !"'] & Z_m \\ \mathrm{Spec}(\mathcal{O}_{Z_m, z}) \arrow[ur, "z = (E, j)"'] & \end{tikzcd}$$

Bijection $\mathrm{Def}(R) = \mathrm{Hom}_W(\mathcal{O}_{Z_m, z}, R)$

Step 1: (Lubin-Tate)

E admits a unique deformation to each $R \in ART$

$$\begin{array}{ccc} \mathrm{Def}_E(R) = \text{single point} & = & \mathrm{Hom}_W(W, R). \\ \cup & & \cup \\ \mathrm{Def}_{(E,j)}(R) = & & \mathrm{Hom}_W(W/p^k W, R) \end{array}$$

where k is the maximal integer s.t. (E, j) lifts to $W/p^k W$.

Step 2: (Gross)

Show that (E, j) lifts to $W/p^k W$

$$\Leftrightarrow k \leq \frac{1 + \mathrm{ord}_p(m)}{2}.$$

Recall: $K = \mathbb{Q}(\sqrt{d_K})$ quadratic imaginary

$$\begin{array}{ccc} \mathfrak{p} & \subseteq & \mathcal{O}_K \\ \text{inert}\,| & & | \\ (p) & \subseteq & \mathbb{Z} \end{array} \qquad \mathbb{F}_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}.$$

$W = W(\overline{\mathbb{F}}_{\mathfrak{p}}) = \hat{\mathcal{O}}^{\text{un}}_{K,\mathfrak{p}}$

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, hook] \arrow[dr] & W \arrow[d] \\ & \overline{\mathbb{F}}_{\mathfrak{p}} = W/\mathfrak{p}W \end{tikzcd}$$

Ring auto.

$$\begin{tikzcd} W \arrow[r, "Fr."] \arrow[d] & W \arrow[d] \\ \overline{\mathbb{F}}_{\mathfrak{p}} \arrow[r, "x \mapsto x^p"] & \overline{\mathbb{F}}_{\mathfrak{p}} \end{tikzcd}$$

ART = category of local Artinian W-algebras w/ residue field $\overline{\mathbb{F}}_{\mathfrak{p}}$.

$E/\overline{\mathbb{F}}_{\mathfrak{p}}$ $\quad \mathcal{O}_K \to \operatorname{End}(E)$ s.t.

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] \arrow[dr] & \operatorname{End}_{\overline{\mathbb{F}}_{\mathfrak{p}}}(\operatorname{Lie}(E)) \\ & \overline{\mathbb{F}}_{\mathfrak{p}} \arrow[u] \end{tikzcd}$$

($\Rightarrow E$ is supersingular)

$\operatorname{Def}_E : \text{ART} \to \text{Sets}$

$$\operatorname{Def}_E(R) = \left\{ \begin{array}{c} \text{CM elliptic curves } \tilde{E} \text{ over } R \\ \text{with } \tilde{E}/\overline{\mathbb{F}}_{\mathfrak{p}} \simeq E \end{array} \right\}$$

Goal: $\operatorname{Def}_E(R)$ = single point

Let $E[p^\infty] = \varinjlim E[p^n]$.

Theorem (Serre-Tate): Let A be any abelian variety over $\overline{\mathbb{F}}_{\mathfrak{p}}$, and $A[p^\infty]$ be its p-divisible group. Then $\forall R \in \text{ART}$ there is a bijection

$$\left\{ \begin{array}{c} \text{deformations of } A \\ \text{to } \tilde{A} \text{ over } R \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{deformations of } A[p^\infty] \\ \text{to } p\text{-divisible grp } G \text{ over } R \end{array} \right\}$$

$$\tilde{A} \longmapsto \tilde{A}[p^\infty]$$

Furthermore, given $j \in End(A)$ and deformation $\tilde{A}$ of A to R, then j lifts to $\tilde{j} \in End(\tilde{A}) \iff$ induced $j \in End(A[p^\infty])$ lifts to $End(\tilde{A}[p^\infty])$.

<u>Def:</u> A covariant <u>Dieudonné module</u> is (M, F, V) where

- M is free W-module of finite rank
- $F, V : M \to M$ additive group homomo. s.t.

$$F(x \cdot m) = x^{Fr} F(m) \quad \forall x \in W$$

$$V(x^{Fr} m) = x V(m)$$

- $F \circ V = p = V \circ F$.

$Lie(M) = M/VM \leftarrow$ vector space over $W/pW = \overline{\mathbb{F}}_p$.

Hodge s.e.s.

$$0 \to VM/pM \to M/pM \to M/VM \to 0.$$

<u>Thm:</u> Equivalence of categories

$$\left\{\begin{matrix} p\text{-divisible groups} \\ \text{over } \overline{\mathbb{F}}_p \end{matrix}\right\} \longleftrightarrow \{\text{Dieudonné modules}\}.$$

$$\left\{\begin{matrix} \text{connected } p\text{-divisible} \\ \text{groups over } \overline{\mathbb{F}}_p \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{D-modules in which} \\ V \text{ is top. nilpotent} \end{matrix}\right\}$$

(i.e. $G(\overline{\mathbb{F}}_p) = 0$
ex. μ_{p^∞} is connected
$E[p^\infty]$ is connected)

Example: 1) $\mathbb{Q}_p/\mathbb{Z}_p \rightsquigarrow M = W$

$$F = p \circ Fr$$
$$V = Fr^{-1}.$$

2) $\mu_{p^\infty} \rightsquigarrow M = W$

$$F = Fr$$
$$V = p\,Fr^{-1}.$$

3) Any s.s. elliptic curve $\rightsquigarrow M_{s.s.} = We_1 \oplus We_2$

$$F = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \circ Fr.$$

$$V = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \circ Fr^{-1}.$$

Hodge s.e.s.

$$0 \longrightarrow \overline{\mathbb{F}}_p e_2 \longrightarrow \overline{\mathbb{F}}_p e_1 \oplus \overline{\mathbb{F}}_p e_2 \longrightarrow \overline{\mathbb{F}}_p e_1 \longrightarrow 0.$$

$$\mathbb{Z}_{p^2} = \{ x \in W \mid x^{Fr^2} = x \}$$

$$\mathrm{End}(M_{s.s.}) = \left\{ \begin{pmatrix} a & pb \\ b^{Fr} & a^{Fr} \end{pmatrix} : a, b \in \mathbb{Z}_{p^2} \right\}$$

= unique max. order in unique quat. alg. over $\mathbb{Q}_p$.

Fix $R \in$ ART. $W(R)$ = witt ring of R (see Joe Rabinoff's notes)

$I_R = \ker(W(R) \twoheadrightarrow R)$.

ring. homom. $W(R) \xrightarrow{Fr} W(R)$

additive group isom. $W(R) \xrightarrow{Ver.} I_R$ $\qquad (x^{ver})^{Fr} = px.$

Def (Zink): a display over R is (P, Q, F, F_1) where (denote the quadruple as just P)

- P is free $W(R)$-module of finite rank.
- Q is a submodule
- $F: P \to P$ additive $\quad F(xm) = x^{Fr} F(m) \quad x \in W(R)$

  $F_1: Q \to P \quad F_1(xm) = x^{Fr} F_1(m)$.
- $I_R P \subset Q$ and P/Q is free over $W(R)/I_R = R$.
- P is generated as a $W(R)$-module by $F_1(Q)$.
- $F_1(x^{ver} m) = x F(m) \quad \forall x \in W(R), m \in P$.

$\mathrm{Lie}(P) = P/Q$.

Hodge s.e.s.

$$0 \to Q/I_R P \to P/I_R P \to P/Q \to 0.$$

<u>Prop.</u>: There is an equivalence of categories

$$\{\text{Dieudonné modules}\} \longleftrightarrow \{\text{displays over } \bar{\mathbb{F}}_p\}$$
$$(M, F, V) \longmapsto (M, VM, F, V^{-1})$$

<u>Def</u>: Given a display P over $R \in ART$, let (M, F, V) be the Dieudonné module of $P/\bar{\mathbb{F}}_p$. Then P is <u>nilpotent</u> if V is top. nilpotent on M.

<u>Thm (Zink)</u>: $\{\text{nilpotent displays}\} \longleftrightarrow \{\text{connected } p\text{-div. groups}\}$.

From now on "display" = "nilpotent display".

A <u>PD-thickening</u> is a surjection $S \to R$ in ART with a divided power structure on $J = \ker(S \to R)$ i.e.,

$$\{\gamma_n : J \to J\}_{n \in \mathbb{Z}^+} \quad \text{s.t. } \gamma_n(x) \text{ "looks like" } \frac{x^n}{n!}.$$

<u>Idea:</u> Use these to define $\exp: J \xrightarrow{\sim} 1+J$

$$x \longmapsto 1 + \sum_{n\geq 1} \gamma_n(x).$$

<u>Examples:</u> 1) If $J \subset p\cdot S$, then canonical divided powers are

$$\gamma_n(p\cdot s) = \underbrace{\frac{p^n}{n!}}_{\substack{\cap \\ p\mathbb{Z}}} s^n$$

in particular $W/p^k W \to W/pW = \overline{\mathbb{F}}_p$ has canonical divided powers.

2) If $J^2 = 0$ then trivial divided powers ("square 0 thickening")

$$\gamma_1(x) = x$$
$$\gamma_n(x) = 0 \quad n > 1.$$

Given any $S \twoheadrightarrow R$ in ART, $J = \ker(S \to R)$, $J^k = 0$ for $k >> 0$

$$S = S/J^k \to S/J^{k-1} \to \cdots \to S/J^2 \to S/J = R$$

each step is $\square$-0 (square 0).

PD thickening $S \to R$, (P, Q) display over R

Hodge s.e.s.

$$0 \to Q/I_R P \to P/I_R P \to P/Q \to 0$$
$$\quad\quad \| \quad\quad\quad\quad \| \quad\quad\quad\quad \|$$
$$0 \to \mathrm{Fil}\, P^{crys}(R) \to P^{crys}(R) \to \mathrm{Lie}(P) \to 0$$

$\exists$ lift (P', Q') of (P, Q) to S. If (P'', Q'') is another lift, then $P'/I_S P' \simeq P''/I_S P''$ (canonically, but depending on PD-thickening)

Call common S-module $P^{crys}(S \to R)$.

- satisfies $P^{crys}(S\to R) \otimes_S R \simeq P^{crys}(R)$.

Two submodules

$$\begin{array}{ccc} & Q'/_{I_S P'} \subset P^{crys}(S\to R) & \text{are equal} \iff \\ \mathcal{L}(P',Q') & \cup\!\shortmid & (P',Q') \simeq (P'',Q''). \\ & Q''/_{I_S P''} = \mathcal{L}(P'',Q'') & \end{array}$$

Def: A lift of the Hodge filtration $\mathrm{Fil}\, P^{crys}(R) \subset P^{crys}(R)$ to S is an S-module direct summand

$$\begin{gathered} \mathcal{L} \subset P^{crys}(S\to R) \\ \Big\downarrow \otimes_S R \\ \mathrm{Fil}\, P^{crys}(R) \subset P^{crys}(R) \end{gathered}$$

Thm (Zink, Groth-Messing): There is a bijection

$$\left\{ \begin{array}{c} \text{lifts of } (P,Q) \\ \text{to } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{lifts of the Hodge filtration} \\ \text{to } S \end{array} \right\}$$

$$(P',Q') \longmapsto \mathcal{L}(P',Q').$$

Example: Go back to E over $\overline{\mathbb{F}}_p$ and fix a PD-thickening $S \to \overline{\mathbb{F}}_p$. Let (P,Q) be the display of $E[p^\infty]$ $(P = M_{s.s.},\ Q = V\cdot M_{s.s})$.

$$\begin{array}{ccccccccc} 0 \to & VM_{s.s}/_{pM_{s.s.}} & \longrightarrow & M_{s.s.}/_{pM_{s.s}} & \longrightarrow & M_{s.s}/_{VM_{ss}} & \to 0 \\ & & & \| & & \| & \\ 0 \to & \mathrm{Fil}\, P^{crys}(\overline{\mathbb{F}}_p) & \longrightarrow & P^{crys}(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Lie}(E) & \to 0 \end{array}$$

Fix a basis e_1, e_2 of $P^{crys}(\overline{\mathbb{F}}_p)$ so that.

$$0 \to \bar{\mathbb{F}}_p e_2 \to \bar{\mathbb{F}}_p e_1 \oplus \bar{\mathbb{F}}_p e_2 \longrightarrow \bar{\mathbb{F}}_p e_1 \to 0$$

We know $P^{crys}(S \to \bar{\mathbb{F}}_p) \otimes_S \bar{\mathbb{F}}_p = P^{crys}(\bar{\mathbb{F}}_p)$. Let $\tilde{e}_1, \tilde{e}_2$ be any lifts of e_1, e_2 so that

$$P^{crys}(S \to \bar{\mathbb{F}}_p) = S\tilde{e}_1 \oplus S\tilde{e}_2.$$

$$\begin{array}{ccc} \mathcal{L} & \subset & S\tilde{e}_1 \oplus S\tilde{e}_2 \\ & & \big\downarrow \\ \bar{\mathbb{F}}_p e_2 & \subset & \bar{\mathbb{F}}_p e_1 \oplus \bar{\mathbb{F}}_p e_2 \end{array}$$

all such $\mathcal{L}$ have the form $S\cdot(t\tilde{e}_1 + \tilde{e}_2)$ with $t \in \underset{\overset{\shortparallel}{I_S}}{m_S} = \ker(S \to \bar{\mathbb{F}}_p)$

$$\left\{\begin{matrix}\text{deformations of } E \text{ to } S \\ \text{(ignoring CM)}\end{matrix}\right\} \xleftrightarrow{S-T} \left\{\begin{matrix}\text{deformations of} \\ E[p^\infty]\end{matrix}\right\} \xleftrightarrow{\text{Zink}} \left\{\begin{matrix}\text{deformations} \\ \text{of } (P,Q)\end{matrix}\right\}$$

$$m_S \longleftrightarrow \{\text{lifts of Hodge filtration}\}$$

Now we need to add the CM.

Thm (Lubin-Tate): Given any $S \in ART$, $\exists!$ deformation of E (with action $\mathcal{O}_K \to End(E)$) to S.

Pf: 1^{st} suppose $S \to \bar{\mathbb{F}}_p$ is a PD-thickening. Let (P,Q) be the a display of E. Recall given $\mathcal{O}_K \to W$,

$$\mathcal{O}_K \otimes_{\mathbb{Z}} W \simeq W \oplus W \qquad \ni e_1 = (1,0)$$
$$x \otimes 1 \longmapsto (x, \bar{x}) \qquad e_2 = (0,1).$$

Given any S-module X with commuting action of $\mathcal{O}_K$ then

$$X = e_1 X \oplus e_2 X$$

summand on which $\mathcal{O}_K$ acts through $\mathcal{O}_K \xrightarrow{x \mapsto x} W \to S$

$\mathcal{O}_K$ acts through $\mathcal{O}_K \xrightarrow{x \mapsto \bar{x}} W \to S$

$\mathcal{O}_K$ acts through $\mathcal{O}_K \to W \to \bar{\mathbb{F}}_p$

$$0 \to \mathrm{Fil}\, P^{crys}(\bar{\mathbb{F}}_p) \to P^{crys}(\bar{\mathbb{F}}_p) \to \mathrm{Lie}(E) \to 0$$

$$P^{crys}(\bar{\mathbb{F}}_p) = e_1 P^{crys}(\bar{\mathbb{F}}_p) \oplus e_2 P^{crys}(\bar{\mathbb{F}}_p)$$

each 1-dimensional

Since $\mathrm{Fil}\, P^{crys}(\bar{\mathbb{F}}_p)$ has dim. 1, $\mathrm{Fil}\, P^{crys}(\bar{\mathbb{F}}_p) = e_i \mathrm{Fil}\, P^{crys}(\bar{\mathbb{F}}_p) \subset e_i P^{crys}(\bar{\mathbb{F}}_p)$ $i=1,2$.

We get equality since both are 1-dim. To see which one, we look at action on $\mathrm{Lie}(E)$ of $\mathcal{O}_K$ and conclude the Hodge S.E.S. is

$$0 \to e_2 P^{crys}(\bar{\mathbb{F}}_p) \to P^{crys}(\bar{\mathbb{F}}_p) \to e_1 P^{crys}(\bar{\mathbb{F}}_p) \to 0.$$

Action of $\mathcal{O}_K$ on (P, Q) induces action of $\mathcal{O}_K \otimes_{\mathbb{Z}} S$ on

$$P^{crys}(S \to \bar{\mathbb{F}}_p) = e_1 - \oplus e_2 -,$$

Deformations of (P,Q) to S $\leftrightarrow$ lifts of Hodge filtration.

$\cup$

Deformations of (P,Q) with $\mathcal{O}_K$-action $\longleftrightarrow$ $\mathcal{O}_K$-stable lifts of Hodge filtration.

$$\begin{array}{ccc} \mathcal{L}^{\otimes}_{S} & \subset & P^{crys}(S \to \overline{\mathbb{F}}_p) \\ & & \Big\downarrow \otimes_S \overline{\mathbb{F}}_p \\ e_2 P^{crys}(\overline{\mathbb{F}}_p) & \subset & P^{crys}(\overline{\mathbb{F}}_p). \end{array}$$

Given $\mathcal{L}$ $\mathcal{O}_K$-stable, $\mathcal{L} = e_1\mathcal{L} \oplus e_2\mathcal{L}$, but $\mathcal{L}$ is free of rank 2 over S $\Rightarrow$ $\mathcal{L} = e_i\mathcal{L} \subset e_i\, P^{crys}(S \to \overline{\mathbb{F}}_p)$

$\Rightarrow$ $\mathcal{L} = e_i\, P^{crys}(S \to \overline{\mathbb{F}}_p)$ $\quad (i=2)$.

For arbitrary $S \twoheadrightarrow \overline{\mathbb{F}}_p$, decompose into PD thickenings and repeat. argument ■

Same set-up as last time.

$z \in Z_m(\overline{\mathbb{F}}_p)$ $\quad z=(E,j)$ $\quad E/\overline{\mathbb{F}}_p$ has CM by $\mathcal{O}_K$ and
$j \in L(E) = \{ j \in \mathrm{End}(E) : j \circ x = \bar{x} \circ j \ \forall x \in \mathcal{O}_K \}$, $\deg(j)=m$.

<u>Thm (Gross):</u> The pair (E,j) lifts to $W_k = W/p^k W$ iff $k \le \frac{1+\mathrm{ord}_p(m)}{2}$.

<u>Proof:</u> Recall $\mathcal{O}_K \hookrightarrow W = \hat{\mathcal{O}}^{unr}_{K,p}$ (since working over $\overline{\mathbb{F}}_p$, so fixes a choice of p).

E supersingular $\Rightarrow$ identify Dieudonné module of E with $M_{s.s.} = We_1 \oplus We_2$ in such a way that induced

$$\begin{aligned}\mathcal{O}_K &\longrightarrow \mathrm{End}(M_{s.s}) = \left\{ \begin{pmatrix} a & pb^{Fr} \\ b & a^{Fr} \end{pmatrix} : a,b \in \mathbb{Z}_{p^2} \subset W \right\} \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{Fr} \end{pmatrix}\end{aligned}$$

Then $j = \begin{pmatrix} & pb \\ b^{Fr} & \end{pmatrix}$ for some $b \in \mathbb{Z}_{p^2}$. We have
$m = \deg(j) = -j^2 = -pbb^{Fr}$. This gives $\frac{1+\mathrm{ord}_p(m)}{2} = 1+\mathrm{ord}_p(b)$.
Recall $\mathcal{O}_K \otimes_{\mathbb{Z}} W = W \times W \ni \varepsilon_1 = (1,0),\ \varepsilon_2 = (0,1)$

$$x \otimes 1 \longmapsto (x, \bar{x}).$$

Then $M_{s.s.} = \varepsilon_1 M_{s.s} \oplus \varepsilon_2 M_{s.s} = We_1 \oplus We_2$. The unique lift E_k of E to W_k corresponds to the unique $\mathcal{O}_K$-stable lift of Hodge filtration. Let P = display of $E[p^\infty]$.

$$\begin{array}{ccc} \varepsilon_2 P^{crys}(W_k \to \overline{\mathbb{F}}_p) & \subset & P^{crys}(W_k \to \overline{\mathbb{F}}_p) \\ & & \big\downarrow \otimes_{W_k} \overline{\mathbb{F}}_p \\ \varepsilon_2 P^{crys}(\overline{\mathbb{F}}_p) & \subset & P^{crys}(\overline{\mathbb{F}}_p). \end{array} \qquad (*)$$

j induces W_k-module endomorphism of $P^{crys}(W_k \to \overline{\mathbb{F}}_p)$.
(E,j) lifts to $(E_k, j_k) \iff j$ preserves $\varepsilon_2 P^{crys}(W_k \to \overline{\mathbb{F}}_p)$.
Fact: $P^{crys}(W_k \to \overline{\mathbb{F}}_p) \cong M_{s.s.} \otimes_W W_k \cong M_{s.s}/p^k M_{s.s.}$
So the diagram $(*)$ becomes

$$\begin{array}{ccc} \mathcal{E}_2\left({}^{M_{ss}}/_{p^k M_{ss}} \right) \subset {}^{M_{ss}}/_{p^k M_{ss}} & & W_k e_2 \subset W_k e_1 \oplus W_k e_2 \\ \Big\downarrow & = & \Big\downarrow \otimes_{W_k} \overline{\mathbb{F}}_p \\ \mathcal{E}_2\left({}^{M_{ss}}/_{p M_{ss}} \right) \subset {}^{M_{ss}}/_{p M_{ss}} & & \overline{\mathbb{F}}_p e_2 \subset \overline{\mathbb{F}}_p e_1 \oplus \overline{\mathbb{F}}_p e_2 . \end{array}$$

j lifts to $j_k \in End(E_k) \iff j(W_k e_2) \subset W_k e_2$.

$\iff j(e_2) = pbe_1 \in W_k e_2 \iff pb = 0$ in W_k

$\iff 1 + ord_p(b) \geq k$. ∎

<u>Cor:</u> length $\mathcal{O}_{Z_{m,z}} = \dfrac{1 + ord_p(m)}{2}$.

Unitary Shimura varieties:

K quad imag. $\hookrightarrow \mathbb{C}$.

$M_{(r,s)}$ = moduli space of (A, λ, κ) over $\mathbb{C}$ where

- A is an abelian variety of dim $r+s$
- $\kappa : \mathcal{O}_K \to End(A)$ s.t. $x \in \mathcal{O}_K$ acts on $Lie(A) \cong \mathbb{C}^{r+s}$ as $x \mapsto \begin{pmatrix} x & & & & \\ & \ddots & & & \\ & & x & & \\ & & & \bar{x} & \\ & & & & \ddots \\ & & & & & \bar{x} \end{pmatrix}$ (r copies of x, s copies of $\bar{x}$.)
- $\lambda : A \to A^\vee$ is a principal polarization s.t. $\forall x \in \mathcal{O}_K$

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\bar{x}"'] & A^\vee \arrow[d, "x^\vee"] \\ A \arrow[r, "\lambda"] & A^\vee . \end{tikzcd}$$

Construction of (A, λ, κ):

Start with a projective $\mathcal{O}_K$-module $\mathfrak{a}$ of rank $r+s$ with perfect Hermitian pairing $\langle , \rangle : \mathfrak{a} \times \mathfrak{a} \to \mathcal{O}_K$

$$\left(\begin{array}{l} \cdot \langle a,b\rangle = \overline{\langle b,a\rangle} \\ \cdot \langle xa,b\rangle = x\langle a,b\rangle = \langle a,\bar{x}b\rangle \end{array} \right)$$

with signature (r,s), i.e., $V = \mathfrak{a} \otimes_{\mathcal{O}_K} \mathbb{C} \cong \mathbb{C}^{r+s}$.

$$\langle a,b\rangle = {}^t a \begin{pmatrix} I_r & \\ & -I_s \end{pmatrix} \bar{b}.$$

Alternating form $\lambda : \mathfrak{a} \times \mathfrak{a} \longrightarrow \mathbb{Z}$

$$\lambda(a,b) = \frac{1}{\sqrt{\operatorname{disc} K}} \left(\langle a,b\rangle - \langle b,a\rangle \right).$$

$(V/\mathfrak{a}, \lambda, \kappa) \notin M_{(r,s)}$ b/c ① $\lambda(ia,a)$ is not pos. def. on V.

κ: obvious $\mathcal{O}_K$ action on V

② $\mathcal{O}_K$ acts on $\operatorname{Lie}(V/\mathfrak{a}) = V$ via $x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}$.

Both of these problems can be fixed by putting a new $\mathbb{C}$-structure on V.

Pick s-plane h s.t. $\langle\, ,\rangle|_h$ is negative definite.

$V = h \oplus h^\perp$. ~~Then~~ The new $\mathbb{C}$-structure is

$$z \cdot^{new} = \begin{cases} z\cdot & \text{on } h^\perp \\ \bar{z}\cdot & \text{on } h. \end{cases}$$

Let V_h be the new $\mathbb{C}$-v.s. Then $\lambda(i \cdot^{new} a, a)$ is pos. def. so defines a Riemann form on $\mathfrak{a}$. $\mathcal{O}_K$ acts as $\mathcal{O}_K \to \mathbb{C}$ on $h^\perp$ and as $\mathcal{O}_K \hookrightarrow \mathbb{C} \xrightarrow{conj} \mathbb{C}$ on h. Thus, it acts as

$$x \longmapsto \begin{pmatrix} x & & & & \\ & \ddots & & & \\ & & x & & \\ & & & \bar{x} & \\ & & & & \ddots \\ & & & & & \bar{x} \end{pmatrix}$$

on $\operatorname{Lie}(V_h/\mathfrak{a}) = V_h$. Thus, $A_h = (V_h/\mathfrak{a}, \lambda, \kappa) \in M_{(r,s)}$.

Let D = space of negative s-planes in V.

$$\underset{\substack{\| \\ G}}{GU(\mathfrak{a} \otimes_{\mathcal{O}_K} K)} \supseteq \Gamma = \{ g \in G(\mathbb{Q}) : g\mathfrak{a} = \mathfrak{a} \} \circlearrowright D$$

(G, D) is a Shimura datum. has dim rs

$$\begin{array}{ccc} \Gamma\backslash D & \longrightarrow & M_{(r,s)} \\ h & \longmapsto & A_h. \end{array}$$

This is not a surjection - only a surjection onto one connected component.

If we take disjoint union over all $(\sigma, \langle , \rangle)$.

__Kudla-Rapaport divisors:__ $n > 1$.

$M_{(1,0)} \times M_{(n-1,1)}$ = ~~moduli~~ moduli of (E, A) CM e.c. ab. var. of dim n w/ $\mathcal{O}_K$-action.

$\mathrm{Hom}_{\mathcal{O}_K}(E, A) \ni f, g$

$$E \xrightarrow{f} A \xrightarrow{\lambda_A} A^\vee \xrightarrow{g^\vee} E^\vee \xrightarrow{\lambda_E^{-1}} E \quad \in \mathrm{End}_{\mathcal{O}_K}(E) = \mathcal{O}_K$$

$\langle f, g \rangle = \lambda_E^{-1} \circ g^\vee \circ \lambda_A \circ f. \in \mathcal{O}_K.$

dim $n-2$

Z_m = moduli of (E, A, j) $\quad E \in M_{(1,0)}$, $A \in M_{(n-1,1)}$,

$\downarrow$ $\qquad j \in \mathrm{Hom}_{\mathcal{O}_K}(E, A)$ s.t. $\langle j, j \rangle = m$.

$M_{(1,0)} \times M_{(n-1,1)}$ $\leftarrow$ dim $n-1$

at such points A isogenous to $E \times (\text{smthg})$

$\langle j, j \rangle$ "measuring degree"

__CM points:__

$$\begin{tikzcd}[column sep=small] & \mathbb{K} \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash, "2"'] & & F \arrow[dl, dash, "n"] \\ & \mathbb{Q} & \end{tikzcd}$$

tot. real Galois

Φ CM type of $\mathbb{K}$ of signature $(n-1, 1)$

$\shortparallel$

$\{\varphi_1, \ldots, \varphi_n\} \subset \mathrm{Hom}(\mathbb{K}, \mathbb{C})$

- $\varphi_1, \ldots, \varphi_n$ have pairwise distinct restrictions to F.
- $\varphi_1|_K, \ldots, \varphi_{n-1}|_K$ are fixed $K \hookrightarrow \mathbb{C}$
- $\varphi_n|_K$ is $K \hookrightarrow \mathbb{C} \xrightarrow{\text{conj}} \mathbb{C}$.

CM_Φ = moduli of (A, λ, κ) A ab. var. of dim n.

$\kappa : \mathcal{O}_{\mathbb{K}} \to \mathrm{End}(A)$ s.t. $x \in \mathcal{O}_{\mathbb{K}}$ acts on $\mathrm{Lie}(A) \cong \mathbb{C}^n$ by

$$x \longmapsto \begin{pmatrix} \varphi_1(x) & & \\ & \ddots & \\ & & \varphi_n(x) \end{pmatrix}.$$

λ = prin. polarization.

The action restricted to $\mathcal{O}_K$ is $x \mapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x & \\ & & & \bar{x} \end{pmatrix}$.

$$\begin{tikzcd} CM_\Phi \arrow[d, "\substack{\text{restrict } \mathcal{O}_{\mathbb{K}} \\ \text{action to } \mathcal{O}_K.}"] \\ M_{(n-1,1)} \end{tikzcd}$$

$\leftarrow$ 0-dim

Set $X = M_{(1,0)} \times M_{(n-1,1)}$, $X^{cm} = M_{(1,0)} \times CM_\Phi$.

X^{cm} is a finite set of points.

$$\begin{tikzcd} Z_m \cap X^{cm} \arrow[r] \arrow[d] & X^{cm} \arrow[d] \\ Z_m \arrow[r] & X \end{tikzcd}$$

<u>Lemma:</u> $Z_m \cap X^{cm} = \phi$.

<u>Proof:</u> a point in the intersection is (E, A, j)

cm by $\mathcal{O}_{\mathbb{K}}$; cm by $\mathcal{O}_K$; $\in \mathrm{Hom}_{\mathcal{O}_K}(E, A)$.

There is an induced K-linear

$$j : \mathrm{Lie}(E) \to \mathrm{Lie}(A).$$

$$\begin{array}{ccc} \mathrm{Lie}(E) & \longrightarrow & \mathrm{Lie}(A) \\ & & \cong\nearrow_{j\otimes 1} \;\longleftarrow\; \mathbb{K} = K \otimes F \text{ linear isom.} \\ & \mathrm{Lie}(E) \otimes_{\mathbb{Q}} F & \end{array}$$

As a K-module $\mathrm{Lie}(E) \otimes_{\mathbb{Q}} F = \mathrm{Lie}(E)^{[F:\mathbb{Q}]=n}$. $\underset{x}{\overset{K}{\Psi}}$ acts as $\begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}$. However, x acts on $\mathrm{Lie}(A)$ by $\begin{pmatrix} x & & & \\ & \ddots & & \\ & & x & \\ & & & \bar{x} \end{pmatrix}$.

Thus, no such (E, A, j) exists. (Moral is they satisfy different signatures in char 0. so they cannot intersect.) $\blacksquare$

$\mathcal{Z}_m$ and $\mathcal{X}$ have integral models over $\mathcal{O}_K$ (Pappas). $\mathcal{X}^{cm}$ has integral model over the reflex field of Φ (since $\mathbb{K}/\mathbb{Q}$ Galois).
$$\underset{\mathcal{O}_{\mathbb{K}}}{\overset{\Phi}{\cap}}$$

All have integral models over $\mathcal{O}_{\mathbb{K}}$. So we have the diagram (script indicates integral models):

$$\begin{array}{ccc} \overset{\dim 0}{\mathcal{Z}_m \cap \mathcal{X}^{cm}} & \longrightarrow & \overset{\dim 1}{\mathcal{X}^{cm}} \\ \downarrow & & \downarrow \\ \underset{\dim n-1}{\mathcal{Z}_m} & \longrightarrow & \underset{\dim n}{\mathcal{X}} \end{array}$$

<u>Theorem (H.):</u> $\underset{\text{as in lecture 1}}{\deg}(\mathcal{Z}_m \cap \mathcal{X}^{cm}) = \dfrac{\#\text{ class grp of } K}{|\mathcal{O}_K^\times|}$

$$\sum_{\substack{\alpha \in F^{\gg 0} \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = n-1}} \sum_{\substack{\mathfrak{p} \subseteq \mathcal{O}_F \\ \text{nonsplit} \\ \text{in } \mathbb{K}}} [1 + \mathrm{ord}_{\mathfrak{p}}(\alpha \underset{\mathrm{Diff}(F/\mathbb{Q})}{\mathcal{D}_F})]$$

$$\cdot \#\{\varpi \in \mathcal{O}_{\mathbb{K}} : \varpi\bar{\varpi} = \alpha \mathfrak{D}_F \mathfrak{p}^{-\varepsilon}\}.$$

where $\varepsilon = \begin{cases} 1 & \mathfrak{p} \text{ inert in } \mathbb{K} \\ 0 & \mathfrak{p} \text{ ram. in } \mathbb{K}. \end{cases}$

An arithmetic divisor on $\mathcal{X}$ is $(\mathcal{Z}, G(z)) = \hat{\mathcal{Z}}$ where $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a divisor and $G(z)$ is a Green function for $\mathcal{Z}(\mathbb{C})$ on $\mathcal{X}(\mathbb{C})$, i.e., smooth $G: \mathcal{X}(\mathbb{C}) - \mathcal{Z}(\mathbb{C}) \longrightarrow \mathbb{R}$ if $f(z) = 0$ is an equation for $\mathcal{Z}(\mathbb{C})$ then $G(z) + \log|f(z)|^2$ is smooth on $\mathcal{X}(\mathbb{C})$.

If f is a rational function on $\mathcal{X}$ then $(\mathrm{div}(f), -\log|f|^2)$ is principal arithmetic divisor of f.

$$\widehat{\mathrm{Pic}}^k(\mathcal{X}) = \text{arith. divisors} / \text{principals.}$$

$$\Big\downarrow \ \widehat{\deg}_{\mathcal{X}^{cm}}(\mathcal{Z}, G) = \deg(\mathcal{Z} \cap \mathcal{X}^{cm}) + \sum_{z \in \mathcal{X}^{cm}(\mathbb{C})} G(z).$$

$$\mathbb{R}$$

There is a natural construction $(\mathcal{Z}_m, G_m(z,v)) = \hat{\mathcal{Z}}_m(v)$ $(v \in \mathbb{R}^+)$ $\in \widehat{\mathrm{Pic}}(\mathcal{X})$

Theorem: There exists a weight 2 Hilbert modular Eisenstein series $\mathcal{E}(\tau, s)$ $\quad \tau \in \underbrace{\mathfrak{H} \times \cdots \times \mathfrak{H}}_{[F:\mathbb{Q}] = n\text{-copies}}$

s.t. 1) $\mathcal{E}(\tau, 0) = 0$

2) Let $\mathcal{F}(\tau, s)$ be the diagonal restriction of $\mathfrak{H} \hookrightarrow \mathfrak{H} \times \cdots \times \mathfrak{H}$, so $\tau \in \mathfrak{H}$.

Fourier expansion

$$\mathcal{F}'(\tau, 0) = \sum_{m=-\infty}^{\infty} c(m, v)\, q^m \qquad \tau = u + iv, \quad q = e^{2\pi i \tau}$$

3) If $m \neq 0$,

$$c(m, v) = \deg_{\mathcal{X}\mathrm{cm}} \hat{Z}_m(v).$$

<u>Conjecture:</u> $\Theta(\tau) = \sum_{m=-\infty}^{\infty} {}^{\mathcal{X}}\hat{Z}_m(v) q^m \in \widehat{\mathrm{Pic}}(\mathcal{X})[[q]]$.

is a vector-valued modular form of wt n.

Given a cuspform f of wt n.

$$\Theta_f = \langle f, \Theta \rangle_{\mathrm{Pet}} \in \widehat{\mathrm{Pic}}(\mathcal{X}) \xrightarrow{\deg_{\mathcal{X}\mathrm{cm}}} \mathbb{R}.$$

$$\begin{aligned} \deg_{\mathcal{X}\mathrm{cm}}(\Theta_f) &= \deg_{\mathcal{X}\mathrm{cm}} \langle f, \Theta \rangle_{\mathrm{Pet}} \\ &= \langle f, \deg_{\mathcal{X}\mathrm{cm}} \Theta \rangle_{\mathrm{Pet}} \\ &= \langle f, {}^{\mathcal{X}}\mathcal{F}'(\cdot, 0) \rangle_{\mathrm{Pet}} \quad \text{(by thm)} \end{aligned}$$

Define an "L"-function $\mathcal{L}(f, s) = \langle f, {}^{\mathcal{X}}\mathcal{F}(\cdot, s) \rangle_{\mathrm{Pet}}$.

$$\deg_{\mathcal{X}\mathrm{cm}} \Theta_f = \mathcal{L}'(f, 0).$$