Growth of Hecke fields in p-adic families and applications:

$E/\mathbb{Q}$ elliptic curve

$\rightsquigarrow$

$$f_E = \sum_{n=1}^{\infty} a(n) q^n \quad \text{on } \Gamma_0(N) \quad wt = 2.$$

$$T_\ell(E) = \varprojlim_n E[\ell^n] \circlearrowleft Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \qquad T_\ell(E) \circlearrowleft Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

$\cong \mathbb{Z}_\ell^2$ write as ρ_ℓ

$$T_\ell E^{I_p} = \begin{cases} 0 & \to 1 - 0 \\ \mathbb{Z}_\ell & \to X - a(p) \\ \mathbb{Z}_\ell^2 & \to \text{unramified} \quad \det(X - \rho_\ell(Frob_p)) \end{cases}$$

$= X^2 - a(p)X + p.$

char polys

$$L(s, E) = \prod_p \det(1 - \rho_\ell(Frob_p) p^{-s})^{-1} = \sum_{n=1}^{\infty} a(n) n^{-s}.$$

$\rho = \{\rho_\lambda\}_{\lambda \text{ prime of } E}$ $\qquad \rho_\lambda \circlearrowleft E_\lambda^2 = V(\rho_\lambda).$

$$L(s, \rho) = \prod_p \det(1 - \rho_\lambda(Frob_p) p^{-s})^{-1} = \sum_{n=1}^{\infty} a(n, f_\rho) n^{-s}$$

$f_\rho \in S_k(\Gamma_0(N), \psi)$, $\det \rho = \mathcal{N}^{k-1} \psi$

cyclotomic character

$K \subset \overline{\mathbb{Q}}$ number field.

$K(a_\rho(n) : n = 1, 2, \dots)$

Fix a prime $p > 2$. Fix N, $p \nmid N$.

Fix ~~an~~ embeddings

$$\mathbb{C} \xleftarrow[i_\infty]{} \overline{\mathbb{Q}} \xrightarrow[i_p]{} \overline{\mathbb{Q}}_p \subset \mathbb{C}_p.$$

$$S_{k+1,\psi} = S_{k+1}(\Gamma_0(Np^{r+1}), \psi)$$
$$r \geq 0$$

$\mathbb{Z}[\psi] \subset \overline{\mathbb{Q}}$ ring generated by $\psi(n)$, $n = 1, 2, \ldots$

$\mathbb{Z}_p[\psi] \subset \overline{\mathbb{Q}}_p$.

$$\mathfrak{h} = \mathbb{Z}[\psi][T(n) \mid n = 1, 2, \ldots] \subseteq \mathrm{End}(S_{k+1,\psi}).$$

$$\mathfrak{h}_{k+1,\psi} = \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{Z}_p[\psi].$$

$T(\ell) = U(\ell)$ if $\ell \mid Np$.

$\mathfrak{h}^{ord}_{k+1,\psi}$ max. ring direct summand of $\mathfrak{h}_{k+1,\psi}$ on which $U(p)$ is invertible.

$$e = \lim_{n \to \infty} U(p)^{n!}$$

then $\mathfrak{h}^{ord}_{k+1,\psi} = e\,\mathfrak{h}_{k+1,\psi} \rightsquigarrow \mathbf{h} = \mathbf{h}_{\psi/\Lambda}$ $\quad \Lambda = \mathbb{Z}_p[[T]]$

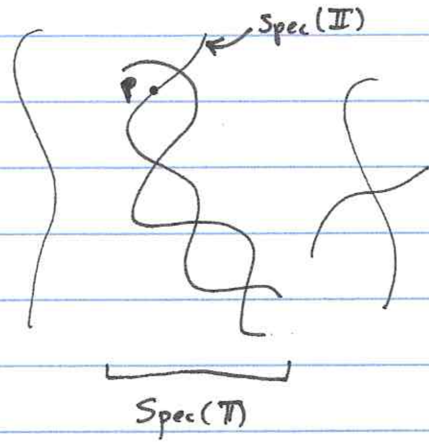
$$\left(1 + p\mathbb{Z}_p = \gamma^{\mathbb{Z}_p} \quad \gamma = 1 + p \right)$$

- $\mathbf{h}$ is free of finite rank over Λ
- equipped with $T(n)$, $U(\ell)$ $\ell \mid Np$.
- $\mathbf{h} / (1+T - \varepsilon(\gamma)\gamma^k)\mathbf{h} \simeq \mathbf{h}^{ord}_{k+1, \varepsilon\psi_k}$ $\quad \forall\, k \geq 1$,

$$\varepsilon : 1+p\mathbb{Z}_p = \Gamma \longrightarrow \mu_{p^\infty}$$

$$\Psi_k = \Psi\,\omega^{1-k}$$

$\mathrm{Spf}(\mathbb{H}) \longleftrightarrow$ rigid analytic space 1-dim



Fix an irred. component Spec ($\mathbb{I}$).

$$\rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{I}).$$

$$\mathrm{Tr}\,\rho_{\mathbb{I}}(\mathrm{Frob}_\ell) = T(\ell)|_{\mathbb{I}} = a(\ell)$$

$\ell \nmid Np$.

$P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ $\quad P : \mathbb{I} \to \overline{\mathbb{Q}}_p$ $\quad \mathbb{Z}_p$-alg. homom.

$$\rho_P = P\circ\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{Q}}_p).$$

This moves the point P around in Spec ($\mathbb{I}$).

We say P is arithmetic if for some $k \geq 1$, $\varepsilon : \Gamma \to \mu_{p^\infty}$ one has $P(1+T-\gamma^k\varepsilon(\gamma)) = 0$.

$k = k(P)$

$\varepsilon = \varepsilon_P$

$$P(T(n)) = P(a(n)) = a_P(n) \in \overline{\mathbb{Q}}_p.$$

$$\Psi_P = \Psi\,\omega^{1-k}\,\varepsilon_P$$

$$f_P = \sum_{n=1}^{\infty} a_P(n) q^n \in S_{k(P)+1}(\Gamma_0(Np^{r(P)+1}), \Psi_P)$$

$$\mathcal{F}_{\mathbb{I}} = \{ f_P : P \text{ arithmetic pt of Spec}(\mathbb{I}) \}$$

$\mathbb{I}$ has (is) CM if $\exists\ G \underset{\text{open}}{\subset} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ s.t.

$\rho_{\mathbb{I}}|_G$ has abelian image.

If $\mathbb{I}$ is CM, $\exists$ M imaginary quadratic field in which $p = \wp\bar{\wp}$
$\uparrow i_p$.

$\exists\ \varphi : \mathrm{Gal}(\overline{\mathbb{Q}}/M) \to \mathbb{I}^\times$ s.t. $\rho_{\mathbb{I}} \simeq \mathrm{Ind}_M^{\mathbb{Q}} \varphi$.

$\rho_P \simeq \mathrm{Ind}_M^{\mathbb{Q}} \varphi_P \qquad \varphi_P = P \circ \varphi \Rightarrow f_P = \Theta(\varphi_P)$

$$= \sum_{\mathfrak{a} \subseteq \mathcal{O}_M} \varphi_P([\mathfrak{a}, M]) q^{N(\mathfrak{a})}$$

~~Measure~~ $K = \mathbb{Q}_p(\mu_{p^\infty}) \qquad K(a_P(p))$ ← $U(p)$ e.v. of f_P.

H-Theorem (Horizontal thm): Take an infinite subset A of arithmetic points in $\mathrm{Spec}(\mathbb{I})$. $\mathcal{H}_A(\mathbb{I}) = K(a_P(p) : P \in A)$ is ~~an~~ a finite extension iff $\mathbb{I}$ has CM.
made of P w/ $k(P) = k$ fixed

Moreover if $\mathbb{I}$ is non-CM,

$$\limsup_{P \in A} [K(a_P(p)) : K] = \infty.$$

"$\Leftarrow$" Galois deformation theory.. this is the easy direction.
We concentrate on the "$\Rightarrow$" direction.

Strategy: $\det(X - \rho_{\mathbb{I}}(\mathrm{Frob}_\ell)) = (X - \alpha_\ell)(X - \beta_\ell)$, $\alpha_\ell, \beta_\ell \in$ quad. ext. of $\mathbb{I}$.

Weil ℓ-number of wt k. $\longrightarrow \alpha_{\ell,P} \in \overline{\mathbb{Q}}$

$K_P = K(\alpha_{\ell,P})$.
| ← bounded deg.

- $\bigcup_{P \in \mathcal{A}} K_P$ contains only finitely many Weil

ℓ-numbers of wt k up to roots of unity.

$$P(\Phi(T)) = \Phi(\varepsilon(\gamma)\gamma^k - 1) \qquad \text{if } P(1+T-\varepsilon(\gamma)\gamma^k) = 0.$$

Change variables and get

$$\alpha_\ell(\varepsilon(\gamma)\gamma^k - 1) = \zeta_\varepsilon \alpha \quad \text{for } \infty\text{'ly many } \varepsilon.$$

change variables again

$$\Phi_\ell(\zeta - 1) = \zeta_\ell$$

$$\downarrow$$

$$\Phi_\ell(1+T) = (1+T)^{s_\ell}$$

$$\downarrow$$

$$\rho_{\mathbb{I}}(\mathrm{Frob}_\ell) \sim \begin{pmatrix} (1+T)^{s_\ell} & 0 \\ 0 & (1+T)^{s_\ell'} \end{pmatrix}$$

$$\begin{tikzcd}
\Lambda = \mathbb{Z}_p[[T]] = \mathbb{Z}_p[[\Gamma]] \arrow[r, "\sigma_2"] & \mathbb{Z}_p[[\Gamma]] = \mathbb{Z}_p[[T]] \\
1+T \longleftrightarrow \gamma \arrow[r, mapsto] \arrow[r, bend right, mapsto] & \gamma^2 \qquad (1+T)^2
\end{tikzcd}$$

$$\underset{\substack{\uparrow \\ \text{rep.}}}{\mathrm{Tr}(\sigma_2 \circ \rho_{\mathbb{I}})} = \underset{\substack{\uparrow \\ \text{not. rep.}}}{\mathrm{Tr}(\rho_{\mathbb{I}}^2)}$$

__Lemma (Bounded degree):__ P arithmetic, $K(f_P) = K(a_P(n) : n = 1, 2, \ldots)$

$$[K(f_P) : K(a_P(p))] < \infty \quad \text{indep. of } P.$$

"Pf": For simplicity, suppose Ψ quadratic.

$$\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K(a_p(p)))$$

$$f_P^\sigma \in S^{ord}_{k+1,\psi_P}$$

$$\mathrm{rk}_{\mathbb{Z}_p[\psi_P]} S^{ord}_{k+1,\psi_P} = \mathrm{rk}_{\mathbb{Z}_p[\psi_P]} \mathbb{H}/(1+T-\gamma^k\varepsilon(\gamma))\mathbb{H} = \mathrm{rk}_\Lambda \mathbb{H}.$$

~~[illegible]~~

Theorem (Honda-Tate):

$$\{\text{isogeny class of } \mathbb{F}_{\ell^k}\text{- simple abelian varieties}\}$$

$$\Big\updownarrow \text{1-1 onto}$$

$$\{\text{Weil } \ell\text{- numbers of wt } k\}/\sim_{conj.}$$

Prop. (Finiteness of Weil numbers): $\mathcal{X}_d$ = set of all ext. of K of degree $= d$ in $\overline{\mathbb{Q}}$ which is tamely ramified at ℓ. We have only finitely many Weil ℓ-numbers of given wt k in $\bigcup_{L\in\mathcal{X}_d} L$ up to equivalence. ($\alpha \sim \beta$ if $\alpha/\beta \in \mu_{p^\infty}$).

$d=1$ $\qquad \alpha \in \mathbb{Q}(\mu_{p^\infty})$ Weil p-number of wt 2

$$\alpha\bar{\alpha} = p \qquad (\alpha) = (p)^{1/2}$$

$$\Rightarrow \alpha \sim \sqrt{(-1)^{\frac{p-1}{2}} p}$$

$$\mathbb{Z}_p^\times \left(\begin{array}{c} \mathbb{Q}(\mu_{p^\infty}) \\ \Big| \\ \mathbb{Q} \end{array} \right) D_\ell = \langle \ell \rangle \quad \text{fin.} \quad (\ell)$$

$$(d) = \prod_p p^{e(p)}$$

$$e(\ell) + e(\bar{\ell}) = k.$$

$$\begin{array}{c} L \\ \Big| \ \ell\text{-tame} \\ K \\ \Big| \\ \mathbb{Q} \end{array} \qquad \left| \{ L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \} / \simeq \right| < \infty$$

as $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ algebras.

R. Robinson conj: $\mathbb{Q}^{ab}$ only finitely many Weil ℓ-numbers of given wt. up to roots of unity.
(Thm of Loxton)

$\Phi \in W[[T]]$ $\quad W$ D.V.R. $/\mathbb{Z}_p$

__Rigidity lemma:__ $\Phi(\zeta - 1) \in \mu_{p^\infty}$ for ∞'ly many $\zeta \in \mu_{p^\infty}$
$\Rightarrow \Phi(T) = (1+T)^s \quad s \in \mathbb{Z}_p$.

$$Z \subset \hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m / W = \mathrm{Spf}(\widehat{W[T, T', T^{-1}, T'^{-1}]}) \qquad \hat{\mathbb{G}}_m = \mathrm{Spf}(\widehat{W[T, T^{-1}]})$$

1-dim formal

completion w.r.t. $(T-1)^{p^n}$

Z is defined by $T = T'^{s}$ $\quad s \in \mathbb{Q}_p$

$T^a = T'^b \quad a, b \in \mathbb{Z}_p$.

Z has onto projection onto 1^{st} factor

Z graph of $\hat{\mathbb{G}}_m \longrightarrow \hat{\mathbb{G}}_m$ $\Rightarrow \Phi(T) = (1+T)^s$

$t \longmapsto \Phi(t-1)$

Regard $\Phi : \hat{\mathbb{G}}_m \longrightarrow \hat{\mathbb{G}}_m$

$$\underset{\overset{\|}{\Phi(\zeta^z - 1)}}{\Phi(\zeta^z)} = \Phi(\zeta^{\sigma_z}) = \Phi(\zeta)^{\sigma_z} = \Phi(\zeta)^z \quad \text{if } \Phi(\zeta) \in \mu_{p^\infty}$$

$z \in \mathbb{Z}_p^\times$

$\overset{\|}{Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})}$ $\qquad \Phi(T)^\sigma \underset{def}{=} \sum_{n=0}^{\infty} a_n^\sigma T^n$

$$\left(\begin{array}{l} \Phi : T \longmapsto \Phi(T) \\ \quad t-1 \longmapsto \underset{\overset{\|}{\Phi(t)}}{\Phi(t-1)} \quad ? \end{array} \right)$$

$\Rightarrow \Phi(t^z) = \Phi(t)^z$ for an open subgrp. $1+p^m\mathbb{Z}_p \subset \Gamma$.

Z graph of Φ is stable under $(t, t') \longmapsto (t^z, t'^z)$.

Pick a point $(t_0, t_0' = \Phi(t_0)) \in Z$ of infinite order in $\hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m$

$$\left(t_0^{1+p^m z}, t_0'^{\,1+p^m z}\right) = (t_0, t_0')\underset{\underset{Z.}{\cap}}{(t_0^{p^m z}, t_0'^{\,p^m z})} \qquad z \in \mathbb{Z}_p$$

(Need to impose $\Phi(a) = 1$)

Z is a root of a formal torus

Z is a formal subtorus.

$\mathrm{Spec}(\mathbb{H}) \supseteq \mathrm{Spec}(\mathbb{I})$ irred. comp. $\quad \rho_{\mathbb{I}}|_G$ has abelian image

$\mathrm{Spec}(\mathbb{H}) \supseteq \mathrm{Spec}(\mathbb{T}) \supseteq \mathrm{Spec}(\mathbb{I})$

$G \underset{\text{open}}{\subset} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$K = \mathbb{Q}(\mu_{p^\infty})$

$\mathbb{I}$ has CM.

<u>H-Theorem:</u> $\mathbb{I}$ is non-CM $\Leftrightarrow$ for any infinite given set $\mathcal{A}$ of arith points, a fixed $k \geq 1$

$$\limsup_{P \in \mathcal{A}} [K(a_P(p)) : K] = \infty$$

Suppose $\limsup_{P \in \mathcal{A}} < \infty \Rightarrow \mathbb{I}$ CM.

$$\begin{array}{c} K(a_P(p), \alpha_\ell(P)) = K_P \\ | \;\; \text{bnd. indep of } P \\ K(a_P(p)) \\ | \;\; \text{bnd. indep of } P \text{ (assumption)} \\ K = \mathbb{Q}(\mu_{p^\infty}) \\ | \\ \mathbb{Q} \end{array}$$

ℓ prime $\quad \rho_{\mathbb{I}}(\mathrm{Frob}_\ell) \sim \begin{pmatrix} \alpha_\ell & \\ & \beta_\ell \end{pmatrix}$

$$\rho_P(\mathrm{Frob}_\ell) \sim \begin{pmatrix} \alpha_\ell(P) & 0 \\ 0 & \beta_\ell(P) \end{pmatrix}$$

<u>Prop. (Eigenvalue formula):</u> $\left(\exp_p(\log_p(\alpha_1)) = \langle \alpha_1 \rangle\right)$ If $\ell >> 0$, then $\exists$ a Weil ℓ-number of wt 1 : α_1 $\quad (|\alpha_1|_p = 1)$

s.t. $\alpha_\ell(P) = \zeta_0 \langle \alpha_1 \rangle^{k(P)} \quad \forall P \in \mathcal{A}$,

in other words
$$\alpha_\ell(T) = \zeta_0 (1+T)^{s_\ell}$$

$$s_\ell = \frac{\log_p(\alpha_1)}{\log_p(\gamma)} \qquad \gamma = 1+p.$$

<u>Pf:</u>

$\bigcup_{P \in \mathcal{A}} K_P$ there are only finitely many Weil ℓ-numbers of wt k up to mult of roots of unity in μ_{p^∞}.

$\mathcal{A} \ni P \longmapsto \alpha_\ell(P)$, replacing $\mathcal{A}$ by its infinite subset.
$\uparrow$
$k(P)=k$

Can assume that $\mathcal{A} \ni P \longmapsto \alpha_1^k \zeta_p$ $\quad \zeta \in \mu_{p^\infty}$.

For simplicity suppose $\mathbb{I} = W[[T]]$.

$\alpha_\ell(1+T-\varepsilon_P(\gamma)\gamma^k) = \zeta_p$

replace ~~by~~ α_ℓ by $\alpha_\ell/\alpha_\ell^k \zeta_{p_0}$ $\qquad T \longmapsto Y = \gamma^{-k}(1+T)-1$.

$\alpha_\ell(\varepsilon_{P_0}(\gamma)-1) = 1$

$\alpha_\ell(\varepsilon_P(\gamma)-1) = \zeta_p \quad \forall P \in \mathcal{A} \rightsquigarrow \alpha_\ell(Y) = (1+Y)^{s_\ell}$

Now go back and compute the exact value of s_ℓ. $\blacksquare$

$\sigma_2 : W[[T]] \cong W[[T]] \qquad (1+T) \longmapsto (1+T)^2 \qquad$ W-alg. auto.

$\qquad \Phi(T) \longmapsto \Phi((1+T)^2-1)$

Ab. image lemma:

$$Tr(\sigma_2 \circ \rho_{\mathbb{I}}|_G) = Tr(\rho_{\mathbb{I}}^2|_G)$$

rep.

$\Rightarrow \rho_{\mathbb{I}}|_G$ has abelian image. (Exercise)

"if" of M-Thm: $\rho = \rho_{\mathbb{I}}$.

$\ell > d$

$$Tr(\rho(Frob_\ell)) = \zeta(1+T)^a + \zeta'(1+T)^{a'}$$

$\downarrow$ specialize $P=(T)$ $\quad W[[T]]/(p) = W/\mathbb{Z}_p$ finite

ζ, ζ' has fin. bdd. orders indep of ℓ.

$\downarrow$

$\bar\rho = \rho \bmod m_{\mathbb{I}}^N + (T) \qquad N >> 0$ to separate ζ, ζ'

$$G = Gal(\bar{\mathbb{Q}}/\bar{\mathbb{Q}}^{\ker(\rho)}) \underset{\text{open}}{\subset} Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \underset{\text{as in lemma}}{\Rightarrow} \blacksquare$$

An abelian variety $A/\mathbb{Q}$ is of $GL(2)$ type if $End^\circ(A/\mathbb{Q}) := End(A/\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} \supset F$ field (CM or tot. real)

s.t. $[F:\mathbb{Q}] = \dim A$.

A has potential CM if $End^\circ(A/\bar{\mathbb{Q}}) \supset M/F$ quad. ext.

A is non-CM if A is not potentially CM.

λ prime in F

$T_\ell A \sim \mathcal{O}_{F,\lambda}^2 \circlearrowleft_{\rho_\lambda} Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ $\quad \rho_A = \{\rho_\lambda\}_\lambda$ strictly compatible system of λ-adic Galois reps.

$$L(s, \rho_A) = \prod_p \det(1 - \rho_\lambda(Frob_p) p^{-s})^{-1}.$$

$A \sim B$ twist equivalent if $L(s, \rho_A \otimes \chi) = L(s, \rho_B)$ where $\chi : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{Q}}^\times$ is a finite order character.

<u>Finiteness Theorem for abelian varieties of $GL(2)$-type:</u>

There are only finitely many twist equiv. classes of non-CM $\mathbb{Q}$-simple ab. var. of $GL(2)$-type with potentially ordinary good reduction at p and good reduction everywhere else.

Khare-Wintenberger/Kisin: $\rho_A \sim f_A \in S_2(\Gamma_0^*(p^r), ?)$

$L(s, \rho_A) = L(s, f_A)$.

f_A generates an auto. rep. $\pi = \bigotimes_\ell \pi_\ell$

Hida
5-18-11
P34

$$\pi_p = \begin{cases} \text{supercuspidal} \rightsquigarrow A \text{ s.s. at } p. \\ \text{ramified prin. series} \rightsquigarrow \text{can twist to } f_P \in \mathcal{J}_{\mathbb{I}} \end{cases}$$

$\mathbb{I} \subset \mathbb{H} \leftarrow$ prime ~~top~~ to p Level 1

Carayol: supercuspidal $\Rightarrow \rho_\lambda|_{D_p} = \mathrm{Ind}_{M_p}^{\mathbb{Q}_p} \varphi$ $\quad M_p/\mathbb{Q}_p$ local quad. field ext.
$\lambda | p$

Frobenius $A \otimes \mathbb{F}_{p^?} = \tilde{A}$ $\circlearrowleft$ ϕ $\qquad \phi^m = \varphi(\text{Artin sym})^{m'}$.

$$\rho_\lambda|_{Gal(\overline{\mathbb{Q}}_p/M_p)} = \varphi \oplus \varphi'.$$

$\varphi'(g) = \varphi(\sigma g \sigma^{-1}) \qquad \sigma \in Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

$\sigma|_{M_p} \neq 1.$

$\varphi'([p, M_p])\varphi([p, M_p]) = p^2$

$\varphi(\sigma[p,M_p]\sigma^{-1}) \longrightarrow \varphi([p,M_p])^2 = p^2 \Rightarrow \tilde{A}$ supersingular.

$\pi_p = \pi(\alpha, \beta)$

$A \otimes \mathbb{F}_{p^?} = \tilde{A}$ $\circlearrowleft$ Φ $\quad$ ord. good reduction $\qquad \mu_p^{\dim A} \hookrightarrow \tilde{A}[p] \downarrow (\mathbb{Z}/p\mathbb{Z})^{\dim A}$

$\Phi = \alpha([p; \mathbb{Q}_p])$

One of α, β, say α is ~~[illegible]~~ unramified.

$$\begin{tikzcd} Gal(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong Gal(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \arrow[r, "\alpha"] & \overline{\mathbb{Q}}^\times \end{tikzcd}$$
fin. order.

α

$\pi \longmapsto \pi \otimes \alpha^{-1}$

f_P.

Replace π by $\pi \otimes \alpha^{-1}$.

$$\underset{\substack{\uparrow \\ \text{unramified}}}{\pi(\alpha, \beta)} \longrightarrow a_P(p) = \alpha([p, \mathbb{Q}_p]) \quad |a_P(p)|_p = 1.$$

$$\Sigma_P = \{ \sigma : \mathbb{Q}(a_P(p)) \longrightarrow \overline{\mathbb{Q}} \mid |a_P(p)^\sigma|_p = 1 \}.$$

A_P has potentially ordinary reduction $\Leftrightarrow$ Σ_P is a CM type of $\mathbb{Q}(a_P(p))$.

<u>Thm (fin. of CM types):</u> On a non-CM family of prime to p level 1 there are only finitely many arithmetic $\underset{\substack{\uparrow \\ \text{wt } 1}}{P}$ s.t. Σ_P is a CM-type of $\mathbb{Q}(a_P(p))$.

<u>Pf:</u> $F_P = \mathbb{Q}(a_P(p))$

$K_P = \mathbb{Q}(\varepsilon_P) \subset \mathbb{Q}(\mu_{p^\infty}) = K$

$L_P = F_P(\varepsilon_P) = \mathbb{Q}(a_P(p), \varepsilon_P)$.

$$\mathrm{Inf}_{L_P} \Sigma_P = \{ \sigma : L_P \to \overline{\mathbb{Q}} \mid \sigma|_{F_P} \in \Sigma_P \}.$$

$A = \{ P \mid k(P) = 1, \Sigma_P \text{ is a CM-type} \}$.

Show if $|A| = \infty \Rightarrow \mathbb{I}$ CM. This will be a contradiction since we started with non-CM.

$\forall P \in A$

$$2 |\mathrm{Inf}_{L_P} \Sigma_P| = [L_P : \mathbb{Q}]$$

Hida
5-18-11
pg 6

$$[K_{\mathfrak{P}} : \mathbb{Q}] = p^{r(\mathfrak{P})-1}(p-1)$$

$$|\mathrm{Inf}_{L_{\mathfrak{P}}} \Sigma_{\mathfrak{P}}| = |\{\sigma \in \mathrm{Inf}_{L_{\mathfrak{P}}} \Sigma_{\mathfrak{P}} : \sigma|_{K_{\mathfrak{P}}} = \sigma_0|_{K_{\mathfrak{P}}}\}| \, [K_{\mathfrak{P}} : \mathbb{Q}]$$

$$= \underbrace{c}_{\text{some const.}} [L_{\mathfrak{P}} : K_{\mathfrak{P}}]\, p^{r(\mathfrak{P})} \qquad \left(\frac{p-1}{p} = c.\right)$$

$\sigma \in \Sigma_{\mathfrak{P}}$ $\quad |a_{\mathfrak{P}}(p)^{\sigma}|_p = 1$

$f_{\mathfrak{P}}^{\sigma} = g_{\mathfrak{Q}}$ another family

$$\begin{tikzcd}[column sep=small]
K(a_{\mathfrak{P}}(p)) \arrow[d, no head] \arrow[dr, no head] & \\
L_{\mathfrak{P}} \subseteq \mathbb{Q}(a_{\mathfrak{P}}(p), \varepsilon_{\mathfrak{P}}) \arrow[dr, no head] & \mathbb{Q}(\mu_{p^\infty}) = K \arrow[d, no head] \\
& \mathbb{Q}(a_{\mathfrak{P}}(p), \varepsilon_{\mathfrak{P}}) \cap K \arrow[d, no head] \\
& K_{\mathfrak{P}} = \mathbb{Q}(\varepsilon_{\mathfrak{P}})
\end{tikzcd}$$

$H = \bigoplus_{j=0}^{p-2} \mathbb{H}_{\omega^j}$ $\;/\Lambda$ free of fin. rk

$$|\mathrm{Inf}_{L_{\mathfrak{P}}} \Sigma_{\mathfrak{P}}| \le \mathrm{rank}_{\mathbb{Z}_p} \frac{H}{((1+T)^{p^n} - \gamma)H}$$

$$= p^{r(\mathfrak{P})}\, \mathrm{rank}_{\Lambda} H$$

$$[K(a_{\mathfrak{P}}(p)) : K] = [L_{\mathfrak{P}} : L_{\mathfrak{P}} \cap K] \le [L_{\mathfrak{P}} : K_{\mathfrak{P}}].$$

Combining all of this gives

$$[K(a_{\mathfrak{P}}(p)) : K] \le c^{-1}\, \mathrm{rank}_{\Lambda} H \quad \forall\, \mathfrak{P} \in A$$

H-Thm $\Rightarrow \mathbb{I}$ has CM. ∎

Hida
5-18-11
p97

$\mathrm{Ord}_p = \{ A_{\underline{p}} : \Sigma_{\underline{p}} \text{ CM-type} \}$

finite set

$p \geq 2$ $\quad p = 3, 5, 7$ no slope 0 family

$\Rightarrow |\mathrm{Ord}_p| = 0.$

$p \geq 11$ $\quad |\mathrm{Ord}_p|$?