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Pg#1

Minimal modularity lifting theorems over imaginary quadratic fields:

(work in progress w/ Frank Calegari)

Introduction:

Let $f \in S_k(\Gamma_0(N), \chi)$ $k \geq 1$ newform., $E_f = \mathbb{Q}(a_n : n \geq 1)$ where $f = \sum a_n q^n$. For each $\wp$ a prime of E_f there exist a continuous irred. Galois rep.

$$r_\wp = r_\wp(f) : G_{\mathbb{Q}} \longrightarrow GL_2(E_{f,\wp})$$

unramified at all $l \nmid Np$ when $\wp | p$, and

$$\det(x - r_\wp(\mathrm{Frob}_l)) = x^2 - a_l x + \chi(l) l^{k-1}.$$

Question: What cont. reps. $r : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ arise in this fashion?

Theorem (Khare-Wintenberger, Kisin): ("Fontaine-Mazur conjecture")

Under some mild hypotheses, r is modular $\Leftrightarrow$

(1) r is unramified at all but finitely many l

(2) $r|_{G_{\mathbb{Q}_p}}$ is "deRham"

(3) $(\det r)(\text{complex. conj}) = -1$ "odd"

deRham?: (1) Let $\varepsilon : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$ be the p-adic cyclotomic char.

This is deRham, in fact, ε^k is deRham $\forall k \in \mathbb{Z}$.

This k is called the Hodge-Tate weight.

(2) Let $\mathbb{I}$ be a Hida family, $P \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ an arithmetic point $\leftrightarrow S_k^{ord}(\Gamma_0(Np^r), \chi)$ $k \geq 2$

then

$$\rho_P|_U \sim \begin{pmatrix} \varepsilon^{k-1} & * \\ 0 & \varepsilon^0 \end{pmatrix} \qquad U \subseteq I_{\mathbb{Q}_p} \text{ (fin. index subgrp of inertia)}$$

is deRham and the Hodge-Tate wts $= \{0, k-1\}$.

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To any n-dimensional deRham rep. of $G_{\mathbb{Q}_p}$ one can associate n integers $(h_1, \ldots, h_n)$ "Hodge-Tate" numbers.

Say r is regular if the h_i are pairwise distinct. For example, f wt k then $r_p(f)$ is regular $\Longleftrightarrow k \geq 2$.

<u>Conjecture (Langlands, Fontaine-Mazur):</u> F number field, $n \geq 1$.

$\exists$ bijections s.t. $L(\pi, s) = L(r, s)$

$$\left\{\begin{array}{c}\text{auto. reps } \pi \text{ of} \\ GL_n/F \\ \text{which are cusp.} \\ \text{and alg.}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{compatible systems of} \\ \text{irred. alg. reps.} \\ r_p : G_F \to GL_n(\overline{\mathbb{Q}}_p)\end{array}\right\}$$

$$\Updownarrow$$

$$\left\{\begin{array}{c}r : G_F \to GL_n(\overline{\mathbb{Q}}_p) \\ \text{irred. alg.}\end{array}\right\}$$

π is algebraic iff infinitesimal char. of π_∞ is alg, i.e., equals that of a fin. dim. irred. rep.

r is algebraic iff (1) $r|_{G_{F_v}}$ is unramified for all but finitely many v

(2) $r|_{G_{F_v}}$ is deRham $\forall\, v|p$.

<u>$F = \mathbb{Q}$, $n = 2$:</u> π cusp. auto. rep.

- $\pi \longleftrightarrow f$ of wt $k \geq 2$ $\quad \{r_p(\pi)\}_p$ exist
- $\pi \longleftrightarrow f$ of wt $k = 1$ $\quad \{r_p(\pi)\}_p$ exist
- $\pi \longleftrightarrow$ Maass eigenforms g $\quad$? Conjecturally these correspond to even reps.

$\qquad \Delta g = \frac{1}{4} g$

Want to discuss method for associating an auto. form to a Galois rep. Most successful method is Taylor-Wiles method.

• F totally real field

$n=2$:

$\pi \longleftrightarrow f$ Hilbert modular forms of wt $\underline{k}=(k_\tau)_{\tau:F\to\mathbb{R}}$.

a) (regular) If all $k_\tau \geq 2$, $r(f)$ were constructed by Carayol, Taylor, ...

If some $k_\tau = 1$, $r(f)$ constructed by Jarvis.

b) F is either CM or totally real, n general

if F is CM, we must assume $\pi\circ c \cong \pi^\vee \otimes \chi$ $\quad c = $ c.c.

if F is tot. real, we must assume $\pi \cong \pi^\vee \otimes \chi$.

If π is regular, then $\{r_p(\pi)\}$ exists (Harris-Taylor, Shin, Chenevier-Harris)

For certain non-regular π, $\{r_p(\pi)\}$ was constructed by Wushi Goldberg. These have HT-weights $h_1 > h_2 > \cdots > h_i = h_{i+1} > h_{i+2} > \cdots > h_n$.

For regular π: $r_p(\pi)$ sits naturally in the étale cohomology of a Shimura variety

$$\pi^\infty \otimes r_p(\pi) \hookrightarrow H^*_{et}(Sh, \mathcal{L})$$

For nonregular π: $r_p(\pi)$ are constructed by finding congruences between π and other regular π''s in the coherent cohomology of Shimura varieties.

c) F quadratic imag. $n=2$, π cusp. reg. algebraic., $\omega_\pi \circ c = \omega_\pi$
(ω_π = central char.) Then $r_p(\pi)$ exists (Taylor's, Harris-Soudry-Taylor)

Automorphic representations in cohomology:

$F = \mathbb{Q}$, $n=2$. f of wt 2, π corresponding auto. rep.

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π contributes to $H^1(X, \mathbb{C})$ (Betti cohom.) and to $H^0(X, \omega^2)$ (coherent cohom.)

f wt 1 $\longleftrightarrow \pi$, π contributes to $H^0(X, \omega)$ and $H^1(X, \omega)$

Serre-duality.

F = quad. imag., $n=2$, π cusp. reg. alg. trivial inf. char.

$$U \subseteq GL_2(\mathbb{A}_F^\infty) \rightsquigarrow X_U = GL_2(F) \backslash G(\mathbb{A}_F) / U \cdot U(2, \mathbb{R}) \mathbb{C}^\times$$

3-dim mnfld.

π contributes to $H^1_{cusp}(X_U, \mathbb{C})$ and $H^2_{cusp}(X_U, \mathbb{C})$

related by Poincare duality.

<u>Galois Theory:</u>

F number field, $\bar{r}: G_F \to GL_2(\bar{\mathbb{F}}_p)$ cts. abs. irred.

Fix
- $\chi: G_F \to \mathbb{Z}_p^\times$ lifts $\det \bar{r}$.
- $S \supset \{v \mid \infty\} \cup \{v \mid p\} \cup \{v : \bar{r}|_{G_{F_v}} \text{ is ramified}\}$

  finite set of places of F.
- For each $\tau: F \hookrightarrow \bar{\mathbb{Q}}_p$ fix $k_{\tau,1} \geq k_{\tau,2} \in \mathbb{Z}$ (Hodge-Tate wts).

(Assume p splits as $v_1 \cdots v_d$ in F. For each v_i, fix $k_{i,1} \geq k_{i,2} \in \mathbb{Z}$, can do this instead of τ's..)

Assume $p \gg 0$.

$$\mathrm{Def}_{\bar{r}}(\underline{k}, S) : ART_{\mathbb{Z}_p} \longrightarrow Sets$$

$$A \longmapsto \left\{ r: G_F \to GL_2(A) \;\middle|\; \begin{array}{l} r \bmod m_A \simeq \bar{r} \\ \det r = (G_F \xrightarrow{\chi} \mathbb{Z}_p^\times \to A^\times) \\ r \text{ unram outside } S \\ r|_{G_{v_i}} \text{ cryst. with H-T.} \\ \text{wts } k_{i,1}, k_{i,2}. \end{array} \right\} \Big/ \sim$$

This is pro-represented by a complete Noetherian local $\mathbb{Z}_p$-alg. $R^{univ}_{\bar{r}}(\underline{k}, S)$.

Since we have fixed weight and level, we expect $R^{univ}_{\bar{r}}(\underline{k}, S)$

to have only finitely many $\bar{\mathbb{Q}}_p$-points.

Write

$$\frac{\mathbb{Z}_p[[x_1, \ldots, x_g]]}{(f_1, \ldots, f_r)} \xrightarrow{\sim} R_{\bar{r}}^{univ}(\underline{k}, S)$$

with g, r minimal. Then Krull-dim $R_{\bar{r}}^{univ}(\underline{k}, S) \geq 1 + g - r$

$\geq 1 - \delta$ using Galois cohom.

where $\delta = \sum_{v|\infty} \dim (\mathrm{ad}_{\xi}^{\circ} \bar{r})^{G_{F_v}} - \sum_{i=1}^{d} (1 - \delta_{k_{i,1} = k_{i,2}})$

Call $\delta = \delta(\underline{k}, \bar{r}|_\infty)$ the "Taylor-Wiles defect".

- $\delta \geq 0$. The Taylor-Wiles method works only for $\delta = 0$.
- δ seems to predict the number of cohomological degrees in which the π's that should correspond to reps r of type $(\underline{k}, S)$ appear. $(= \delta + 1)$

<u>Example:</u> 1) ~~[illegible]~~ $F =$ tot. real field, $k_{i,1} > k_{i,2}$ $\forall i$, $\bar{r}$ is tot. odd.

[In general $\dim (\mathrm{ad}^{\circ} \bar{r})^{G_{F_v}} = \begin{cases} 3 & \text{if } v \text{ complex} \\ 3 & \text{if } v \text{ real, } \det(c_v) = 1 \\ 1 & \text{if } v \text{ real, } \det(c_v) = -1 \end{cases}$]

$$\delta = \underbrace{1 + \cdots + 1}_{[F:\mathbb{Q}]} - (1 + \cdots + 1) = 0$$

$\delta = 0 \iff$ these conditions hold.

① 2) $F = \mathbb{Q}$ $\bar{r} =$ odd, $k_1 = k_2 = 0$ ($\leftrightarrow$ wt 1 forms)

$\delta = 1 - 0 = 1$

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② 3) F quad imag., $k_{i,1} > k_{i,2} = (1,0)$ for $i=1,2$ ($\Leftrightarrow$ π reg. alg. triv. inf. char)

$$\delta = 3-(1+1) = 1.$$

Crystalline reps. $r: G_{\mathbb{Q}} \longrightarrow GSp_4(\overline{\mathbb{Q}}_p)$ odd similitude char.

H.T. weights $(h_1 \geq h_2 \geq h_3 \geq h_4)$ $h_1+h_4 = h_2+h_3$.

- regular case $\delta = 0$ π's appear in one degree (Betti, coh.)

③ $\cdot\, (h,h,0,0)$ $h>0$ $\delta = 1$ π's appear in $H^0(X, W_{\lambda(h)})$ and $H^1(X, W_{\lambda(h)})$.

④ $\cdot\, (h,0,0,-h)$ $h>0$ $\delta=1$ π's appear in $H^1(X, W_{\lambda(h)})$ and $H^2(X, W_{\lambda(h)})$

duality

<u>Main Result:</u>

Suppose we are in one of the 4 cases ①–④ where $\delta=1$. To be concrete, ②.

F quad. imag. $\quad r: G_F \longrightarrow GL_2(\mathbb{Q}_p)$

p splits as v_1, v_2

- unram. a.e.
- crystalline HT wts $(1,0)$ at v_1, v_2
- $\bar{r}|_{G_F(\zeta_p)}$ irred
- r is minimal deformation of $\bar{r}$.

Suppose following conditions hold:

<u>I. (Serre's conjecture)</u>: $U \subset GL_2(\mathbb{A}_F^\infty)$ cmpct open, let $\mathbb{T}(U) \subset H^2(X_U, \mathbb{Z}_p)$ $\mathbb{Z}_p$-alg. gen. by T_v at good primes v. Let $\mathfrak{m} = (p, T_v - F(\mathrm{Frob}_v)$ v good place$) \triangleleft \mathbb{T}(U)$. Then $\mathfrak{m}$ is a proper maximal ideal of $\mathbb{T}(U^{\min})$.

<u>II. (Existence of Galois reps)</u>: $\forall\, U \subset U^{\min}$ $\exists$ $r_U^{\mathrm{mod}}: G_F \to GL_2(\mathbb{T}(U)_{\mathfrak{m}})$ satisfying nice properties.

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<u>III (Cohomology vanishing integrally):</u> $H^i(X_u, \mathbb{Z}_p)_m = (0) \quad i \neq 1,2.$

<u>IV (Duality):</u> $\dim H^1(X_u, \mathbb{Q}_p)_m = \dim H^2(X_u, \mathbb{Q}_p)_m$

Then $R_{\bar{r}}^{univ, min} \cong \mathbb{T}(U^{min})_m$ and act freely on $H^2(X_{U^{min}}, \mathbb{Z}_p)_m$.