Periods and heights of Heegner point

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Selmer groups, L-functions, and Galois deformations

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F = totally real field, $\mathbb{A} := \mathbb{A}_F$

π = cuspidal representation of $GL_2(\mathbb{A})$

E/F = quadratic extension

$\chi : E^\times \backslash \mathbb{A}_E^\times \longrightarrow \mathbb{C}^\times$

$\omega_\pi \, \chi \big|_{\mathbb{A}_F^\times} = \begin{cases} 1 \\ \eta_{E/F} \end{cases}$ quadratic character for E/F (will not discuss this case)

$L(s, \pi, \chi)$ = Rankin-Selberg L-series

$L(s, \pi, \chi) = L(1-s, \pi, \chi)\, \varepsilon(s, \pi, \chi)$, $\varepsilon(1/2, \pi, \chi) = \pm 1$.

Set $\varepsilon(1/2, \pi, \chi) = \prod \varepsilon_v(1/2, \pi_v, \chi_v) = (-1)^{\#\Sigma}$

$\Sigma = \{ v$ place of F, $\varepsilon_v(1/2, \pi_v, \chi_v) \neq \omega, \eta_v(-1) \}$ finite set

<u>Question:</u> Find a formula for $L(1/2, \pi, \chi)$ if Σ is even and $L'(1/2, \pi, \chi)$ if Σ is odd.

<u>Even case:</u> Σ = even

B = quaternion alg. $/F$ with ramification set Σ

$G' = B^\times$ as an algebraic group $/F$.

π' = Jacquet-Langlands correspondence of π on $G'(\mathbb{A})$.

There is an embedding $E \hookrightarrow B$.

$E^\times$ as a subgroup of G'.

$f \in \pi'$, then define

$$\ell(f,\chi) = \int_{\mathbb{A}^\times E^\times \backslash E^\times_{\mathbb{A}}} f(g)\,\chi(g)\,dg$$

Assume π = unitary.

$\underline{\text{Thm (Waldspurger, at least when } \omega_\pi = 1)}$: Assume that $f = \otimes_v f_v \in \pi'$ is decomposable. Then

$$\frac{|\ell(f,\chi)|^2}{\langle f,f\rangle} = \frac{\zeta_F(2)\, L(1/2,\pi,\chi)}{2\, L(1,\pi,ad)} \cdot \prod_v c(f_v,\chi_v)$$

with
$$c(f_v,\chi_v) = \frac{L(1,\eta_v)\, L(1,\pi_v,ad)}{\zeta_v(2)\, L(1/2,\pi_v,\chi_v)} \int_{F_v^\times \backslash E_v^\times} \frac{\langle \pi_v'(t) f_v, f_v\rangle}{\langle f_v,f_v\rangle}\,\chi_v(t)\,dt.$$

$\langle f,f\rangle$ = Petersson product wrt Tamagawa measure and

$\langle f_v,f_v\rangle$ = any Hermitian form on π_v.

$c(f_v,\chi_v) \neq 0 \quad \forall\, v, \quad c(f_v,\chi_v) = 1$ for almost all v.

Remarks: For newform f, there is a more precise formula proved by Gross, Zhang under some assumption in ramification.

Thus, the even case is completely solved, we have a nice formula here.

Odd case: Σ = odd. In this case we know very little in general. Assume $\pi\otimes\chi$ is motivic.

(1) $\Sigma \supseteq$ all arch. places. We can conjecture:

$L'(\tfrac{1}{2}, f, \chi) =$ Height of Heegner cycles in some local system on a Shimura curve of $U(1,1)$.

$$\pi \otimes \chi \longleftrightarrow H^1(X, \mathcal{F})$$

(2) $\Sigma \not\supseteq$ all arch. places. This case we know virtually nothing.

Assume now that π is discrete of wt $(2, \ldots, 2)$ at arch. places, χ is a finite character. Then $\Sigma \supseteq$ all arch. places.

$B_{/\mathbb{A}}$ = quaternion alg $/\mathbb{A}$ with ramification set Σ.

For each $U \leq B_{/\mathbb{A}_f}^{\times}$ open compact, X_U = Shimura curve$/_F$.

$$X = \varprojlim X_U \circlearrowleft B_{/\mathbb{A}_f}^{\times}.$$

For $\tau : F \hookrightarrow \mathbb{C}$ a complex place, $B(\tau)_{/F}$ quat. alg with ram. set $\Sigma \setminus \{\tau\}$

$$X_{U,\tau}(\mathbb{C}) = B(\tau)^{\times} \backslash \mathfrak{h}^{\pm} \times \hat{B}^{\times} / U,$$

$$\widehat{B(\tau)} \simeq \hat{B} := B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$$

$E_{/\mathbb{A}} \hookrightarrow B_{/\mathbb{A}}$ $\rightsquigarrow$ defines a morphism $C \to X_E$

$C = \varprojlim C_U$ scheme of zero dim. $/E$

$$C(\bar{E})_U = E^{\times} \backslash \hat{E}^{\times} / \hat{E}^{\times} \cap U$$

$\circlearrowleft$ $\mathrm{Gal}(\bar{E}/E)$ by class field theory.

$$C_{\tau_E}(\mathbb{C}) = \{(z_0, t) \mid z_0 \in \mathfrak{h} \text{ fixed by } E,\ t \in \hat{F}\}$$

Assume $U \cap \hat{E}^\times \subset \operatorname{Ker}(\chi : \hat{E}^\times \to \mathbb{C}^\times)$. Then define

$$Y_{U,\chi} = \operatorname{vol}(U) \sum_{t \in E^\times \backslash \hat{E}^\times / U \cap \hat{E}^\times} \chi(t) \big([z_0, t] - \underset{\substack{\uparrow \\ \text{Hodge class}}}{\xi_t} \big) \in \operatorname{Jac}^\circ(X_U)$$

$\{Y_{U,\chi}\}$ = projective system of divisors on X.

$f \in \pi'$, define an endomorphism

$$T_{f \otimes \bar{f}} : \pi' \longrightarrow \pi'$$
$$h \longmapsto (h, \bar{f}) \cdot f$$

This is a Hecke operator. It also acts on $\operatorname{Jac}(X_U) \otimes_{\mathbb{Q}} \mathbb{C}$.

<u>Thm (Yuan-W. Zhang-S. Zhang):</u> Assume that $f = \otimes f_v$ decomposable.

$$\frac{\langle Y_\chi, T_{f\otimes\bar{f}} Y_\chi \rangle_{NT}}{\langle f, f \rangle} = \frac{\xi_F(2) L'(1/2, \pi, \chi)}{4 L(1, \pi, \mathrm{ad})} \prod_v C(f_v, \chi_v).$$

<u>Remarks:</u> • $F = \mathbb{Q}$, $\Sigma = \{\infty\}$ (Heegner condition) then

Thm = GZ under the conditions that $\omega_\pi = 1$, χ unram.

and a few other conditions.

• For general F, $\omega_\pi = 1$, S.Z. Annals (2002), Asian J. (2003), MSRI (2004) under assumptions on ramifications.

- B. Howard More general cases

$$\pi_v \otimes \chi_v$$
$$\updownarrow$$
$$\pi_v \otimes \rho\circ\det \otimes \chi_v \otimes \rho^{-1}\circ N.$$

Applications: 1) Apply Euler system to get bound on $A_\chi(K)$

A_χ = abelian variety $/F$

$L(s, A_\chi) = \prod L(s-\tfrac{1}{2}, \pi^\sigma, \chi^\sigma)$ $(\pi^\sigma, \chi^\sigma)$ - conjugates of (π, χ).

2) $A = \mathbb{Q}$ - curve

$A/_K$ K quadratic

$A \neq CM$ $\mathrm{rk}\, A(K) = 0, 1$ if $\mathrm{ord}_{s=1} L(s, A) = 0$ or 1.

Linear forms:

$G = GL_2 \times E^\times/\Delta(F^\times)$ unitary group

$(\pi, \chi) \rightsquigarrow \Pi$ on $G(\mathbb{A})$

$T = E^\times/F^\times \hookrightarrow G$ diagonally

Thm: (Tunnell, Waldspurger, Saito):

1) $v \in \Sigma$ iff $\mathrm{Hom}_{T_v}(\Pi_v, \mathbb{C}) = 0$,

2) $\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) \leq 1$

$$G'_{/\mathbb{A}} = B^\times_{/\mathbb{A}} \times E^\times_{/\mathbb{A}} / \Delta(/\mathbb{A}^\times).$$

π' = Jacquet-Langlands on $G'_{/\mathbb{A}}$, (π', χ).

$G'_{/\mathbb{A}}$ is the unique ? of $G_{/\mathbb{A}}$ s.t. $\mathrm{Hom}(\pi, \mathbb{C}) \neq 0$.

$$\dim \mathrm{Hom}_{T \times T}(\pi' \otimes \tilde{\pi}', \mathbb{C}) = 1.$$

We will construct 3-linear forms α, β, γ.

Waldspurger formula $\quad \beta = \alpha\, L(1/2, \pi)$

G Z $\quad \gamma = \alpha\, L'(1/2, \pi)$.

1) Construction of α

$f_v \in \pi'_v, \quad \tilde{f}_v \in \tilde{\pi}'_v$

$$\int_{T_v^\times} (\pi'_v(t) f_v, \tilde{f}_v)\, dt$$

In the spherical case, $= \dfrac{\zeta_v(2)\, L(1/2, \pi_v)}{L(1, \eta_v)\, L(1, \pi_v, \mathrm{ad})} = A(\pi_v) \neq 0.$

Define
$$\alpha_v(f_v, \tilde{f}_v) = A(\pi_v)^{-1} \int_{T_v^\times} (\pi'_v(t) f_v, \tilde{f}_v)\, dt$$

$$\alpha = \prod \alpha_v \in \mathrm{Hom}_{T_{/\mathbb{A}} \times T_{/\mathbb{A}}} (\pi' \otimes \tilde{\pi}', \mathbb{C})$$

$$\neq 0$$

<u>Construction of β:</u> $(\Sigma = \text{even})$

$$B_{/\mathbb{A}} = B \otimes \mathbb{A}, \quad G'_{/\mathbb{A}} = G'(\mathbb{A})$$

$$\pi' \subset \mathcal{A}(G'(F) \backslash G'(\mathbb{A}))$$

$$f, f' \in \pi'$$

$$\beta(f, f') = \int_{T(F) \backslash T(\mathbb{A})} f(t)\,dt \int_{T(F) \backslash T(\mathbb{A})} \tilde{f}(t)\,dt$$

<u>Waldspurger formula:</u>

$$\beta = * \, L(1/2, \pi)\, \alpha$$

<u>Construction of γ:</u> $(\Sigma = \text{odd})$

Shimura curve X by $G'_{/\mathbb{A}}$

$$T_{/\mathbb{A}} \hookrightarrow G'_{/\mathbb{A}}$$

$$\rightsquigarrow \begin{array}{c} Y \longrightarrow X_E \\ | \\ \text{Shimura variety for } T \end{array}$$

$$Y_{\xi} \in \varprojlim_{U} \mathrm{Jac}(X_U)$$

$$f, \tilde{f} \in \pi \otimes \pi'$$

$T_{f\otimes\tilde f}$

$$\gamma(f,\tilde f) := \langle Y_\xi , \; T_{f\otimes\tilde f}\, Y_\xi \rangle_{NT}$$

$$\in \mathrm{Hom}_{T(\mathbb{A})}(\pi' \otimes \tilde\pi', \mathbb{C})$$

<u>GZ</u>: $\gamma(f,\tilde f) = L'(1/2, \pi)\, \alpha(f,\tilde f)$

<u>Kernel functions:</u>

$$\mathcal{S}(B_{\mathbb{A}} \times \mathbb{A}^\times) \otimes \mathcal{A}(E_{\mathbb{A}}^\times / E^\times) \ni \phi$$

$\circlearrowright$ $B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times \times GL_2(\mathbb{A}) \times E_{\mathbb{A}}^\times$

fix an additive character ψ of $F \backslash \mathbb{A}$

$$\tilde\phi(x,u,t) = \int_{\mathbb{A}^\times} \phi(xz, uz^{-2}, tz)\, dz$$

$$\tilde{\mathcal{S}} = \{ \tilde\phi \quad \forall \phi \} \circlearrowleft \underset{t_1}{G'_{\mathbb{A}}} \times \underset{t_2}{G'_{\mathbb{A}}} \times \underset{t_3}{G_{\mathbb{A}}}$$

on $\mathcal{A}(E_{\mathbb{A}}^\times / E^\times)$ by $t_1^{-1} t_2 t_3$.

<u>Shimizu lifting:</u> For any cuspidal $\pi \xrightarrow{\varphi} \mathbb{C}$

$$\dim \mathrm{Hom}(\tilde{\mathcal{S}}, \tilde\pi' \otimes \pi' \otimes \pi) = 1$$

Let $\Theta \in \mathrm{Hom}(\tilde{\mathcal{S}}, \tilde\pi \otimes \pi' \otimes \pi)$, $\varphi \in \pi'$

$$\langle \varphi, \mathcal{F}\Theta(\phi) \rangle = \frac{\zeta(2)}{L(1, \pi, ad)} \int_{N(\mathbb{A})\backslash GL_2(\mathbb{A})} W_\varphi(g)\, \omega(g)\phi(1,1,1)\, dg.$$

$\mathcal{A}_\Sigma(G)$ = cusp forms π with local sgn set Σ

$\mathcal{A}_\Sigma(G' \times G')$ = Θ-lifts of π in $\mathcal{A}_\Sigma(G)$.

space of kernel forms

$$\mathrm{Hom}_{T(\mathbb{A}) \times T(\mathbb{A})}(\mathcal{A}_\Sigma(G' \times G'), \mathbb{C}) \hookrightarrow \mathrm{Hom}_{T^2 \times G}(\underbrace{\widetilde{\mathcal{S}}}_{\text{space of kernel forms}}, \mathcal{A}_\Sigma(G)).$$

$$\underset{\alpha,\,\beta,\,\gamma}{\Psi}$$

$\alpha \longleftrightarrow$ Mixed Θ - Eisenstein series

$$\mathcal{E} = \sum_{\gamma \in P \backslash G} \delta(\gamma g)^s \sum_{(x,u) \in E \times F^\times} u(\gamma g)\,\phi(x,u)$$

$$\beta \longleftrightarrow \Theta_\phi(h,g) = \sum_{x,u} \omega(h,g)\,\phi(x,u)$$

$$\gamma \longleftrightarrow \mathcal{Z}_\phi = \langle Z_\phi,\ Y \times Y \rangle$$

$Y \times Y$ $\cap$ Zero cycle on $X \times X$

$$Z_\phi = \sum_{(x,y,t) \in K \backslash B_{/\mathbb{A}} \times /\mathbb{A}^\times \times E^\times_{/\mathbb{A}}} \phi(x,y,t)\, Z(x,y,t) \in CH^1(X \times X)$$

linear comb. of Hecke ops.

$\Theta = \otimes \Theta_v$

$\frac{1}{2}\mathcal{E} = \Theta \longleftrightarrow$ Waldspurger

$\frac{1}{2}\mathcal{E}' = \mathcal{Z}_\phi \longleftrightarrow$ GZ.