Construction of p-adic L-functions for

unitary groups II

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Selmer groups, L-functions, and Galois

deformations

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Joint work w/ Michael Harris & J.S. Li Skinner

goal (modest): Construct p-adic L-functions (families) of auto. forms on unitary groups

This talk: Explain this construction in a simple setting.

- work with imaginary quadratic field
- stick to definite unitary groups
- work with simple automorphy types

Situation:

F : imaginary quadratic field

p : prime that splits in F

$F \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C} \simeq \mathbb{C}_p$

This picks out $v | p$ in F.

$\rho : G_F \to GL_n(\overline{\mathbb{Q}}_p)$ continuous, geometric

(PC) $w | p$, $\rho|_{D_w} \simeq \begin{pmatrix} \sigma_w^- & * \\ 0 & \sigma_w^+ \end{pmatrix}$ σ_w^- has Hodge Tate weights < 0, σ_w^+ has HT wts ≥ 0.

$$\dim \sigma_w^- = \dim \sigma_{\bar{w}}^+ .$$

Expectation (Coates, Perrin-Riou, ...) The values $L(\rho \otimes \chi, 0)$ are interpolated by a p-adic L-function.

F_∞ / F max. $\mathbb{Z}_p$-ext.

$\Gamma_F = \mathrm{Gal}(F_\infty / F) \simeq \mathbb{Z}_p^2$.

χ finite character of Γ_F.

More precisely, there should exist

$$\mathcal{L} = \mathcal{L}_p \in \mathcal{O}[[\Gamma_F]] \qquad (\mathcal{O} \text{ sufficiently large } p\text{-adic ring})$$

s.t.

remove p^{th} E.f.

$$\mathcal{L}(\chi) = (*) \prod_{w|p} \frac{L(0, \sigma_{\bar{w}}^{-} \chi_w)}{L(1, (\sigma_{\bar{w}}^{-} \chi_w)^\vee)} \; L^p(\rho \otimes \chi, 0)$$

periods, powers of p,
Gamma factors, etc.

Examples:

① $n = 1$ $\quad \Psi$ Hecke character of $\mathbb{A}_F^\times$, $\Psi_\infty(z) = z^k$, $k > 0$.

$\sigma_\Psi : G_F \to \overline{\mathbb{Q}}_p^\times$, $L(\sigma_\Psi, s) = L(s, \Psi)$

at v : HT wt $= -k$

$\bar{v}$: HT wt $= 0$.

One would expect a p-adic L-function interpolating

$$L^p(0, \Psi\chi)\,(1 - \Psi_v \chi_v(\varpi_v))^{-1}\,(1 - \Psi_v^{-1}\chi_v^{-1}(\varpi_v) p^{-1})$$

This was constructed by Katz.

② f wt 2 eigenform,

ρ_f usual p-adic rep. associated to f HT wt 0 and 1.

Ψ as in example ①.

(P_c) satisfied in two cases:

(a) $k = 0$ $\quad \rho = \rho_f \otimes \sigma_\Psi \varepsilon$ $\quad L(f, \Psi, 1)$
ord at p.

(b) $k \geq 2$ $\quad \rho = \rho_f \otimes \sigma_\Psi$ $\quad L(f, \Psi, 0)$

p-adic L-functions are constructed by Hida, Perrin-Riou, ...

③ W definite Hermitian space over F of dim n.

eg $\langle x, y \rangle = \pm x \cdot {}^t\bar{y}$.

$G = U(W)$ unitary group over $\mathbb{Q}$.

π cuspidal auto. rep. of $G(\mathbb{A})$ with trivial minimal K_∞-type.

One expects (and often knows) that there exists a Galois representation

$$\rho_\pi : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$$

s.t. • HT wts $v, \bar{v}$: $0, 1, \ldots, n-1$

• $\rho_\pi \circ c \simeq \rho_\pi^\vee \otimes \varepsilon^{n-1}$

• $L(\rho_\pi, s) = L(\pi, s - \frac{n-1}{2}, st)$

ψ as in Example ① with $k \geq n$.,

$$\rho = \rho_\pi \otimes \sigma_\psi$$

HT wts : at v : $-k, 1-k, \ldots, n-1-k < 0$

$\bar{v}$: $0, 1, \ldots, n-1 \geq 0$

p-adic L-function should interpolate

$$(*) \quad \frac{L_v(0, \rho_\pi \otimes \psi\chi)}{L_v(1, (\rho_\pi \otimes \psi\chi)^\vee)} \underset{\substack{\| \\ L^p(\pi, \psi\chi, \frac{k-n}{2})}}{L^p(0, \rho_\pi \otimes \psi\chi)}$$

This subsumes Examples ① and ②b.

Key ingredient in construction of p-adic L-function:

"doubling method"

$2W = W \oplus -W$ signature (n,n) = sg (n)

$H = U(2W)$ $(U(n,n))$ $G \times G \hookrightarrow H$

$W^d \subset 2W$ diagonal

$$\begin{array}{l} P \subseteq H \\ \| \\ \mathrm{Stab}(W^d) \end{array}$$

(More generally, one should look at $sg(W) = (\dim \sigma_v^+, \dim \sigma_v^-)$.)

$P = MN$ *Levi decomp.*

$M \xrightarrow{\sim} GL(W^d)$

χ unitary Hecke char. of $\mathbb{A}_F^\times$, $\chi_\infty = z^\kappa |z|^{-\kappa}$.

$f \in \mathrm{Ind}_P^H (\chi \cdot \delta^s)$

$f(h,s)$ $\delta(g) = |\det m|_F$ where $g = mnk \in MNK_{\mathbb{A}}$

$$E_f(h,s) = E_f(h,\chi,s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash H(\mathbb{Q})} f(\gamma h)$$

π cusp. rep of $G(\mathbb{A})$ $(U(W))$

$\varphi \in \pi$, $\varphi' \in \pi^\vee$, $\varphi'_\chi = \varphi' \otimes \chi^{-1}$, $\varphi = \otimes \varphi_v$ $\varphi'_v = \otimes \varphi'_v$, $f = \otimes f_v$

$\varphi'(g)\chi^{-1}(\det g)$

$$Z(s,\varphi,\varphi',f,\chi) = \int_{(G\times G)(\mathbb{Q}) \backslash (G\times G)(\mathbb{A})} E_f((g,g'))\varphi(g)\varphi'_\chi(g')\,dg\,dg'$$

$$Z(s) = \prod_v \underbrace{Z_v(s)}$$

$$= \int_{G(\mathbb{Q}_v)} f_v((g,1)) \langle \pi(g)\varphi_v, \varphi'_v \rangle_v \, dg.$$

If everything is unramified

$$Z_v(s) = L_v(\pi, \chi, s+\tfrac{1}{2}, St) \langle \varphi_v, \varphi'_v \rangle_v$$

<u>Problem</u>: dealing with the ramified cases..

Take $S \supseteq \{\infty, p, \ell$ at which π, χ or G is ramified, $\ell \mid \mathrm{disc}\, F\}$

For $\ell \notin S$, choose all the data to be unramified.

For $\ell \neq p, \infty$, can choose f_ℓ to have small support and so $Z_v(s) =$ constant (volume term)

For $\ell = \infty$, for simple minimal K_∞-types can choose a good f_∞ (Harris, Garrett, J.S. Li)

For $\ell = p$, Michael Harris explained a good choice of f_p.

"good":

1st talk: local Fourier coeff. has a simple form, interpolated p-adically by inspection.

2nd talk: the local zeta integrals $Z_v(s)$ using this section have the correct shape (computations carried out by J.S. Li)

Strategy for constructing $\mathcal{L}_{\rho_\pi \otimes \psi}$:

This strategy goes back to Katz.

Geometric part: ψ', $\psi'_\infty = z^k |z|^{-k}$, $\psi = \psi' |\cdot|_F^{-k/2}$

① Eisenstein measure $\chi \longmapsto \mathbb{E}(\chi) = E_f(h, \psi'\chi, \frac{k-n}{2})$ measure on $\Gamma_F = 1 + p\mathcal{O}_{F,p}$ taking values in space of $\underbrace{p\text{-adic modular forms}}_{\text{Hida}}$ for H.

② restrict to p-adic modular forms on $G \times G$

③ "pair" against forms on $G \times G$ related to π

Automorphic part:

reinterpret pairing as the zeta integral (this is where periods show up)

One can carry this out, get the p-adic L-function $\mathcal{L}_{\rho_\pi \otimes \sigma_\psi}$.

Can allow π to vary (as long as (PC) is preserved)

- Hida family

W - definite Hermitian space

$L_p \subset W_p$ self-dual lattice (assumption)

$\|$

$L_v \oplus L_{\bar{v}}$. L_v defines a model of $G/\mathbb{Z}_p$.

$$\begin{array}{ccc}
G/\mathbb{Q}_p \subseteq GL(W_p) = GL(W_v) \times GL(W_{\bar{v}}) & & \\
\searrow^{\sim} & \downarrow & \\
& GL(W_v) & \\
& \cup\!\mid & \\
G/\mathbb{Z}_p \longleftrightarrow & GL(L_v)/\mathbb{Z}_p = G_v & \\
& \mathrel{\rotatebox{90}{$\cong$}} & \\
& GL_n/\mathbb{Z}_p &
\end{array}$$

standard
$B \subset GL(L_v) = G/\mathbb{Z}_p$

standard parabolic
$Q \supset B \longleftrightarrow n = n_1 + \cdots + n_\ell$

$$Q = \begin{pmatrix} \boxed{}_{n_1}^{n_1} & & * \\ & \ddots & \\ 0 & & \boxed{}_{n_\ell}^{n_\ell} \end{pmatrix}$$

$\|$

$M_Q N_Q$

$$g_1, \ldots, g_\ell \longmapsto (\det g_1, \ldots, \det g_\ell)$$

$$M_Q \xrightarrow{\sim} GL_{n_1} \times \cdots \times GL_{n_\ell} \longrightarrow \underbrace{\mathbb{Z}_p^\times \times \cdots \times \mathbb{Z}_p^\times}_{\ell}$$

Automorphic forms on G

χ alg. char. of $GL_n/\mathbb{Z}$

$$\begin{array}{ccccc} G(\mathbb{R}) & \subset & G(\mathbb{C}) = G(\mathbb{C}_p) & \supset & G(\mathbb{Q}_p) \\ & \nwarrow & & \nearrow & \\ & & G(\mathbb{Q}) & & \end{array}$$

$K \subset G(\mathbb{A}_f)$ open compact

$\nu : K \longrightarrow \mathbb{C}^\times$ character

$$A(\chi, K, \nu) = \{ f : G(\mathbb{A}) \longrightarrow \mathbb{C} \quad \text{s.t.} \quad f(\gamma g k_\infty k) = \chi(k_\infty)\nu(k) f(g) \quad \forall\ \gamma \in G(\mathbb{Q}),\ k_\infty \in G(\mathbb{R}),\ k \in K \}$$

Shift the action to p

$$A(\chi, K, \nu) \xrightarrow{\ \sim\ } A_p(\chi, K, \nu)$$

$$\| $$

$$\{ f : G(\mathbb{A}_f) \to \mathbb{C}_p : \quad f(\gamma g k) = \chi(k_p)\nu(k) f(g) \quad \forall\ \gamma \in G(\mathbb{Q}),\ k \in K \}$$

$$f \longmapsto (g \mapsto \chi(g_p) f(g)).$$

Suppose $K = G(\mathbb{Z}_p)K^p$

$$K_Q(m) = \{ k \in K : k_p \bmod p^m \in Q'(\mathbb{Z}/p^m\mathbb{Z}) \}$$

$$A_Q^{p\text{-adic}}(K^p, R) = \varprojlim_r \varinjlim_m A_p(K_Q(m), R/p^r)$$

$R = p$-adic ring

action of $M_Q(\mathbb{Z}_p) \simeq GL_{n_1}(\mathbb{Z}_p) \times \cdots \times GL_{n_\ell}(\mathbb{Z}_p)$ acting through $M_Q(\mathbb{Z}_p) \to \underbrace{\mathbb{Z}_p^\times \times \cdots \times \mathbb{Z}_p^\times}_{\ell}$

$\underline{\nu} = (\nu_1, \ldots, \nu_d)$ where we allow this to vary.

$A_{\mathbb{Q}}^{p\text{-adic}}[\underline{\nu}]$