Ikeda's Conjecture on the Period of the

Ikeda Lifting

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Selmer groups, L-functions, and Galois

deformations

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Ikeda's conjecture on the period of the Ikeda lifting : Kawamura

This is joint work with H. Katsurada.

§1 Introduction:

Let $k \in \mathbb{Z}_{>0}$, $S_k(\Gamma) = \mathbb{C}$ - v.s. of cusp forms of wt k level Γ.

$\Gamma = SL_2(\mathbb{Z})$: $f \in S_k(SL_2(\mathbb{Z}))$ a normalized Hecke eigenform with

Fourier expansion $f(z) = \sum_{N \geq 1} a(N) \exp(2\pi\sqrt{-1}Nz)$.

Let χ be a Dirichlet character. Define

$$L(s,f,\chi) = \prod_p \left[\left(1 - \chi(p)\alpha_p p^{\frac{(k-1)}{2}-s}\right)\left(1 - \chi(p)\alpha_p^{-1} p^{\frac{k-1}{2}-s}\right)\right]^{-1}$$

where $\alpha_p, \alpha_p^{-1} \in \mathbb{C}$ s.t. $\alpha_p + \alpha_p^{-1} = a(p)\, p^{\frac{-(k-1)}{2}}$, and

$$L(s,f,Ad,\chi) = \prod_p \left[(1-\chi(p)p^{-s})(1-\chi(p)\alpha_p^2 p^{-s})(1-\chi(p)\alpha_p^{-2}p^{-s})\right]^{-1}.$$

In particular, if χ is a principal character then we write

$L(s,f)$ and $L(s,f,Ad)$.

The Petersson product of f is defined by

$$\langle f,f\rangle = \int_{SL_2(\mathbb{Z})\backslash \mathfrak{H}} |f(z)|^2 y^k \frac{dx\,dy}{y^2}.$$

Remark (Sturm): If $1 \leq m \leq k-1$ and $\chi(-1) = (-1)^{m-1}$

then $$\frac{L(m,f,Ad,\chi)}{\pi^{k+2m-1}\langle f,f\rangle} \in \mathbb{Q}(f,\chi)$$

$\Gamma = SL_2(\mathcal{O}_F)$: $F/\mathbb{Q}$ real quadratic

For simplicity we assume the narrow class number of F is 1.

Let $\hat{f} \in S_k(SL_2(\mathcal{O}_F))$ be the Doi-Naganuma lift of $f \in S_k(SL_2(\mathbb{Z}))$. Then

$$L(s,\hat{f}) = L(s,f)\, L(s,f,\chi_F).$$

Fact 1: (Doi-Hida-Ishii):

$$(*) \quad \frac{2^{2k}\langle \hat{f},\hat{f}\rangle}{D_F^k \langle f,f\rangle^2} = \frac{L(1,f,Ad,\chi_F)}{\pi^{k+1}\langle f,f\rangle} \in \mathbb{Q}(f)_{+}$$

Doi-Hida-Ishii Conjecture: Let K be a sufficiently large # field and $\mathfrak{p}$ a prime ideal in K.

Then the following two conditions are equivalent

① $\mathfrak{p} \mid$ (Numerator of $(*)$)

② $\exists\, g \in S_k(SL_2(\mathcal{O}_F))$ not coming from Doi-Naganuma lifting s.t. $g \equiv \hat{f} \pmod{\mathfrak{p}}$.

Remark: ① $\Rightarrow$ ② is known to hold under some assumptions. (Urban, Katsurada)

The aim of this talk is to consider the analogy of these results in the Siegel modular case. We will discuss the Petersson inner product of the Ikeda lifting with itself. We will then discuss the applications of the period result to congruences between Siegel modular forms.

§2. Ikeda lift and Ikeda conjecture:

Let $n \in \mathbb{Z}_{>0}$. Define

$$Sp_n(\mathbb{Z}) = \{ \gamma \in GL_{2n}(\mathbb{Z}) : {}^t\gamma J \gamma = J \}$$

where $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Let $F \in S_k(Sp_n(\mathbb{Z}))$ a Hecke eigenform.

Attached to F is the standard L-fctn:

$$L(s, F, St) = \prod_p \left[(1-p^{-s}) \prod_{i=1}^{n} (1-\beta_p^{(i)} p^{-s})(1-(\beta_p^{(i)})^{-1} p^{-s}) \right]^{-1}$$

where $\{\beta_p^{(i)}\}_{1 \le i \le n}$ are the Satake parameters of F. The Petersson product is defined by

$$\langle F, F \rangle = \int_{Sp_n(\mathbb{Z}) \backslash \mathfrak{h}_n} |F(Z)|^2 \det(Y)^k \frac{dX\,dY}{\det Y^{n+1}}$$

where $\mathfrak{h}_n = \{ Z \in Sym_n(\mathbb{C}) \mid Im(Z) > 0 \}$.

Remark (Böcherer, Mizumoto):

$$Sym_n^*(\mathbb{Z})_+ = \{ T \in Sym_n(\mathbb{Q}) \mid 2T \text{ pos. def. even integral} \}.$$

$$\forall\, T \in Sym_n^*(\mathbb{Z})_+ \Rightarrow \forall\, 1 \le m \le k-n,\ m \equiv 0 \pmod 2, \text{ then}$$

$$\frac{L(m, F, St)\,|A_F(T)|^2}{\pi^{\text{some power}} \langle F, F \rangle} \in \mathbb{Q}(F).$$

Fix $n, k \in 2\mathbb{Z}_{>0}$ s.t $k > n+1$, $f \in S_{2k-n}(SL_2(\mathbb{Z}))$ a normalized Hecke eigenform. Then $\exists$ $I_n(f) \in S_k(Sp_n(\mathbb{Z}))$ a Hecke eigenform s.t.

$$L(s, I_n(f), st) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, f).$$

This is a result of Ikeda.

<u>Remark:</u> (1) $n=2$ $I_2(f)$ = Saito-Kurokawa lift of f

(2) $I_n(f)$ can be constructed by $\tilde{f} \in S_{k-\frac{n-1}{2}}(\Gamma_0(4))$ where $\tilde{f}$ is a Hecke eigenform corresponding to f via the Shimura correspondence.

For $\{\beta_p^{(i)}\}$ the Satake parameters of $I_n(f)$, then

$$\beta_p^{(i)} = \begin{cases} \alpha_p \, p^{i-1/2} & \text{if } 1 \le i \le n/2 \\ \alpha_p^{-1} \, p^{i-\frac{(n+1)}{2}} & \text{o/w} \end{cases}.$$

Thus, $|\beta_p^{(i)}| \neq 1$, so $I_n(f)$ is an example of a counterexample to Ramanujan conjecture for $Sp_n(\mathbb{Z})$.

<u>Question:</u> $\langle I_n(f), I_n(f) \rangle = ?$

$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$

$\tilde{\zeta}(s) = \Gamma_{\mathbb{C}}(s)\zeta(s)$

$\Lambda(s, f) = \Gamma_{\mathbb{C}}(s) L(s, f)$.

$$\tilde{\Lambda}(s,f,Ad) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k+n-1)\,L(s,f,Ad).$$

Remark: • $\tilde{\xi}(2i) = \dfrac{|B_{2i}|}{2i} \in \mathbb{Q}^\times$.

• for $1 \le i \le k - n/2 - 1$ then $\dfrac{\tilde{\Lambda}(2i-1,f,Ad)}{\langle f,f\rangle} \in \mathbb{Q}(f)$.

Ikeda Conjecture:
$$\frac{\langle f,f\rangle\langle I_n(f), I_n(f)\rangle}{\langle \tilde{f},\tilde{f}\rangle} = 2^{-\alpha(n,k)}\Lambda(k,f)\prod_{i=1}^{n/2}\tilde{\Lambda}(2i-1,f,Ad)\tilde{\xi}(2i)$$

where $\alpha(n,k) = (n-1)(k-n/2+1) \in \mathbb{Z}$.

Main Thm (Katsurada-K): Ikeda's conjecture is true for every $n \in 2\mathbb{Z}_{>0}$.

Cor: $\forall$ $D<0$ fund. disc

$$\frac{\langle I_n(f), I_n(f)\rangle}{\langle f,f\rangle^{n/2}} = \frac{2^{-\beta(n,k)}\,a_{\tilde{f}}(|D|)^2\,\Lambda(k,f)\,\tilde{\xi}(n)}{|D|^{k-\frac{n+1}{2}}\,\tilde{\Lambda}(k-n/2,f,\chi_D)}\cdot\prod_{i=1}^{n/2}\tilde{\Lambda}(2i-1,f,Ad)\tilde{\xi}(2i).$$

Pf: Use Kohnen-Zagier result. ∎