Heegner points, Iwasawa theory, and the Gross-Zagier theorem

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Selmer groups, L-functions, and Galois deformations

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$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\mu_{p^\infty}) \arrow[d, dash, "\Delta"] \\
C_\infty \arrow[r, dash] \arrow[d, dash] & \mathbb{Q}_\infty \arrow[d, dash] \\
K \arrow[r, dash] & \mathbb{Q}
\end{tikzcd}$$

$$\mathrm{Gal}(C_\infty/K) \xrightarrow[\sim]{\varepsilon_{cycl}} 1 + p\mathbb{Z}_p$$

$$\Lambda(C_\infty) = \mathbb{Z}_p[\![\mathrm{Gal}(C_\infty/K)]\!]$$

<u>Thm (Perrin-Riou):</u> There exists $\mathcal{L}(C_\infty) \in \Lambda(C_\infty)$ s.t.

$\forall\ \chi : \mathrm{Gal}(C_\infty/K) \longrightarrow \overline{\mathbb{Q}}_p^\times$ of finite order,

$$\begin{tikzcd}
\mathcal{L}(C_\infty) \arrow[d, mapsto] & \in & \Lambda(C_\infty) \arrow[d, "\chi"] \\
L(E/K, \chi, 1) & \in & \overline{\mathbb{Q}}_p
\end{tikzcd}$$

up to simple factors

$$\left(\text{Fix} \quad \begin{tikzcd}[row sep=small] & \overline{\mathbb{Q}}_p \\ \overline{\mathbb{Q}} \arrow[ur] \arrow[dr] & \\ & \mathbb{C} \end{tikzcd} \right)$$

For $s \in \mathbb{Z}_p$, define $L_p(E/K, s)$ by

$$\begin{tikzcd}
\mathcal{L}(C_\infty) \arrow[d, mapsto] & \in & \Lambda(C_\infty) \arrow[d, "\varepsilon_{cycl}^{s}"] \\
L_p(E/K, s) & \in & \overline{\mathbb{Q}}_p
\end{tikzcd}$$

$L_p(E/K, 0) = L(E/K, 1) = 0$ since $L(E/K, s) = -\left(\frac{N}{\mathrm{disc}\,K}\right) L(E/K, 2-s)$

$= -1\ L(E/K, 2-s)$

$\Rightarrow\ L(E/K, 1) = 0.$

Thm (Perrin-Riou): Assume p split in K. Then

$$\langle z, z \rangle = \frac{d}{ds} L_p(E/K, s)\Big|_{s=0}$$

up to simple factors.

Anticyclotomic variation:

Let $\mathcal{O}_{p^s} = \mathbb{Z} + p^s \mathcal{O}_K$.

H_{p^s} = ring class field of $\mathcal{O}_{p^s}$

$$\mathrm{Gal}(H_{p^s}/K) \simeq \hat{K}^\times / K^\times \hat{\mathcal{O}}_{p^s}^\times$$

$$\simeq \mathrm{Pic}(\mathcal{O}_{p^s}).$$

Fact: $\mathrm{Gal}(H_{p^s}/\mathbb{Q})$ is a generalized dihedral group.

Fact: $H_{p^\infty} = \bigcup H_{p^s}$ contains a unique $\mathbb{Z}_p$-extension of K

$$\begin{tikzcd}
H_{p^\infty} \arrow[d, dash, "\text{finite}"] \\
D_\infty \arrow[d, dash] \\
K
\end{tikzcd}$$

$$\begin{tikzcd}
& K_\infty \arrow[dl, dash] \arrow[dr, dash] & \\
D_\infty \arrow[dr, dash] & & C_\infty \arrow[dl, dash] \\
& K &
\end{tikzcd}$$

$$\begin{array}{ccc} \mathbb{C}/\mathcal{O}_{p^s} \xrightarrow{h_s} \mathbb{C}/(\mathcal{O}_{p^s}\cap \mathfrak{n})^{-1} & \in & X_0(N)(H_{p^s}) \\ & & \downarrow \\ & & E(H_{p^s}) \\ & & \downarrow \\ & h_s \in & \mathrm{Sel}(H_{p^s}, Ta_p E) \end{array}$$

Let $\alpha \in \mathbb{Z}_p^\times$ be unit root of $X^2 - a_p(E)X + p$. Then

$$\frac{1}{\alpha^s} h_s - \frac{1}{\alpha^{s+1}} h_{s-1} \in \mathrm{Sel}(H_{p^s}, Ta_p E).$$

are norm compatible. Take the limit and then

$\mathrm{Norm}_{H_{p^\infty}/D_\infty}$ to get

$$Z_\infty \in S_\infty = \varprojlim \mathrm{Sel}(D_s, Ta_p E)$$

Set $\Lambda(D_\infty) = \mathbb{Z}_p[\![\mathrm{Gal}(D_\infty/K)]\!]$. There exists

$\Lambda(D_\infty)$-adic height pairing

$$\langle\ ,\ \rangle_\infty : S_\infty \times S_\infty \longrightarrow \Lambda(D_\infty).$$

Perrin-Riou construct $\mathcal{L}(K_\infty) \in \Lambda(K_\infty) = \mathbb{Z}_p[\![\mathrm{Gal}(K_\infty/K)]\!]$.

Identify $\Lambda(K_\infty) \simeq \Lambda(D_\infty)[\![\mathrm{Gal}(C_\infty/K)]\!]$

$$\simeq \Lambda(D_\infty)[\![T]\!]$$

Expand $\mathcal{L}(K_\infty) = \cancel{\mathcal{L}(D_\infty)} + \mathcal{L}'(D_\infty)T + \mathcal{L}''(D_\infty)T^2 + \cdots$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{0}$ because of functional equation

<u>Thm: (Howard):</u> Assume p split in K. Then

$$\langle z_\infty, z_\infty \rangle_\infty = \mathcal{L}'(D_\infty).$$

<u>Hida Theory:</u>

Let $\{f_\alpha\}$ be the Hida family containing the (p-stabilization of) the cusp form of E, i.e., there is a finite flat $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$-algebra R where each α is a "classical point", $\alpha \in \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(R, \overline{\mathbb{Q}}_p)$, each $f_\alpha \in S_?(\Gamma_0(N) \cap \Gamma_1(p^?), \chi_\alpha, \overline{\mathbb{Q}})$.

Only consider f_α of even wt. For such f_α, χ_α has a square root. Let $f_\alpha^* = f_\alpha \otimes \chi_\alpha^{-1/2}$ trivial character.

Hida constructs rank 2 R-module with $G_{\mathbb{Q}}$-action T

s.t. $\quad T \otimes_{\alpha: R \to \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p = V(f_\alpha) =$ Galois representation associated to f_α.

$$
\begin{tikzcd}
\Theta : G_{\mathbb{Q}} \arrow[r, "\varepsilon_{cycl}"] & \mathbb{Z}_p^\times \simeq \mu_{p-1} \times (1+p\mathbb{Z}_p) \arrow[d] \\
& 1+p\mathbb{Z}_p \arrow[d, "\gamma \mapsto \gamma^{1/2}"] \\
& 1+p\mathbb{Z}_p \arrow[d, hook] \\
& \mathbb{Z}_p[[1+p\mathbb{Z}_p]]^\times \arrow[d, hook] \\
& R^\times
\end{tikzcd}
$$

(exists since $p \neq 2$)

Set $T^* = T \otimes \Theta^{-1}$. Then $V(f_\alpha^*) = T^* \underset{\alpha : R \to \overline{\mathbb{Q}}_p}{\otimes} \overline{\mathbb{Q}}_p$.

For some $\alpha \in T_pE \otimes \overline{\mathbb{Q}}_p \simeq V(f_\alpha^*)$, so $\exists$ $T^* \longrightarrow T_pE$.

<u>Thm (Howard)</u>: There exists a canonical class $Z^{big} \in Sel(K, T^*)$

$$
\begin{tikzcd}
Z^{big} \in Sel(K,T^*) \arrow[d, mapsto] \\
Z \in Sel(K, T_pE).
\end{tikzcd}
$$

For any f_α^*, let Z_α be the image of Z^{bis} under

$$Sel(K, T^*) \longrightarrow Sel(K, V(f_\alpha^*)).$$

Thm (Howard): Suppose f_α^* has weight 2. Then

$$Z_\alpha \neq 0 \iff \operatorname*{ord}_{s=1} L(f_\alpha^*/K, s) = 1.$$

What is Z_α? Suppose $f_\alpha \in S_2(\Gamma_0(N) \cap \Gamma_1(p^s), \chi_\alpha, \overline{\mathbb{Q}})$.

$$\begin{array}{ccc}
\mathbb{C}/\mathcal{O}_{p^s} & \longrightarrow & \mathbb{C}/(\mathcal{O}_{p^s} \cap \mathfrak{n})^{-1} \\
\Big\downarrow \begin{array}{l}\text{cyclic } p^s\text{-isogeny} \\ \text{pick generator} \\ \text{of kernel}\end{array} & & \in X(\underbrace{\Gamma_0(N) \cap \Gamma_1(p^s)}_{\Gamma_s})(K(p^s)) \\
\mathbb{C}/\mathcal{O}_K & & \Big\downarrow \\
 & & J(\Gamma_s)(K(p^s)) \\
 & & \Big\downarrow \\
 & & \mathrm{Sel}(K(p^s), Ta_p J(\Gamma_s)) \\
 & & \Big\downarrow \\
 & & \mathrm{Sel}(K(p^s), V(f_\alpha)) \\
 & & \Big\downarrow \text{project onto } \chi^{-1/2} \text{ part} \\
 & & \mathrm{Sel}(K(p^s), V(f_\alpha)) \\
 & & \wr\!\! \mid \\
\begin{array}{c} Z_\alpha \\ \cap \\ \mathrm{Sel}(K, V(f_\alpha^*)) \end{array} & \xrightarrow{\text{res}} & \mathrm{Sel}(K(p^s), V(f_\alpha^*))
\end{array}$$

Fact: The sign in the functional equation of f_α^* is constant with only finitely many exceptions. Let $w = \pm 1$ be the generic value.

Conjecture (Greenberg):

1) Suppose $w=1$. Then

at center pt of funct. eq.

ⓐ $\operatorname{ord} L(f_\alpha^*, s) = 0$ for a.e. f_α^*

ⓑ $\dim \mathrm{Sel}(\mathbb{Q}, V(f_\alpha^*)) = 0$ for a.e. f_α^*.

2) Suppose $w=-1$. Then

ⓐ $\operatorname{ord} L(f_\alpha^*, s) = 1$ for a.e. f_α^*

ⓑ $\dim \mathrm{Sel}(\mathbb{Q}, V(f_\alpha^*)) = 1$ for a.e. f_α^*.

Thm (Kato, Kitagawa, Mazur): Suppose $w=1$ and $\exists\, f_\beta^*$ s.t. $\operatorname{ord} L(f_\beta^*, s) = 0$, then 1) ⓐ and 1) ⓑ hold.

Thm (Howard): Suppose $w=-1$. Suppose $\exists\, f_\beta^*$ of weight 2 s.t. $\operatorname{ord} L(f_\beta^*, s) = 1$, then 2) ⓑ holds. and $\operatorname{ord} L(f_\alpha^*, s) = 1$ for a.e. f_α^* of wt 2.

Sketch of proof:

$\operatorname{ord} L(f_\beta^*, s) = 1$

$\Rightarrow$ one can choose K s.t. $\operatorname{ord} L(f_{\beta/K}^*, s) = 1$.

$\Rightarrow Z_\beta \neq 0 \Rightarrow Z^{bis} \neq 0$.

$\Rightarrow \mathrm{Sel}(K, T^*) \cong R$ by Euler systems

$w=-1 \Rightarrow Z^{bis}$ fixed by $\mathrm{Gal}(K/\mathbb{Q})$

$\Rightarrow \mathrm{Sel}(\mathbb{Q}, T^*) \cong R$

$\Rightarrow$ ⓑ.

$Z^{big} \neq 0 \Rightarrow Z_\alpha \neq 0$ for a.e f_α^*.

$\Rightarrow \mathrm{ord}\, L(f_\alpha^*/K, s) = 1$ for a.e. f_α^* of wt 2.

$\Rightarrow \mathrm{ord}\, L(f_\alpha^*, s) = 1$.

$\square$