

Heegner points, class field theory, and the

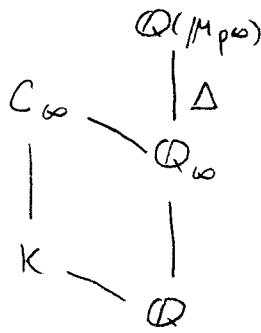
Gross-Zagier theorem

Ben Howard

Selmer groups, L -functions, and Galois
deformations

UCLA

3-27-2008



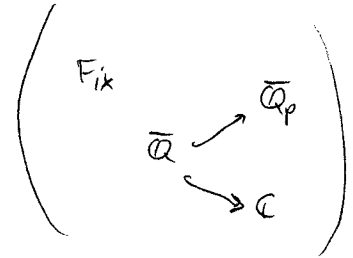
$$\text{Gal}(C_\infty/K) \xrightarrow{\text{cycl}} 1+p\mathbb{Z}_p$$

$$\Lambda(C_\infty) = \mathbb{Z}_p \llbracket \text{Gal}(C_\infty/K) \rrbracket$$

Thm (Perrin-Riou): There exists $\mathcal{L}(C_\infty) \in \Lambda(C_\infty)$ s.t.

$\forall \chi: \text{Gal}(C_\infty/K) \rightarrow \overline{\mathbb{Q}_p}^\times$ of finite order,

$$\begin{array}{ccc}
 \mathcal{L}(C_\infty) \in \Lambda(C_\infty) & & \\
 \downarrow & & \downarrow \chi \\
 L(E/K, \chi, 1) \in \overline{\mathbb{Q}_p} & & \\
 \text{up to simple} & & \\
 \text{factors} & &
 \end{array}$$



For $s \in \mathbb{Z}_p$, define $L_p(E/K, s)$ by

$$\begin{array}{ccc}
 \mathcal{L}(C_\infty) \in \Lambda(C_\infty) & & \\
 \downarrow & & \downarrow \varepsilon_{\text{cycl}}^s \\
 L_p(E/K, s) \in \overline{\mathbb{Q}_p} & &
 \end{array}$$

$$\begin{aligned}
 L_p(E/K, 0) = L(E/K, 1) = 0 & \text{ since } L(E/K, s) = -\left(\frac{N}{\text{disc } K}\right) L(E/K, 2-s) \\
 & = -1 L(E/K, 2-s)
 \end{aligned}$$

$$\Rightarrow L(E/K, 1) = 0.$$

Thm (Perrin-Riou): Assume p splits in K . Then

$$\langle z, z \rangle = \frac{d}{ds} L_p(E/K, s) \Big|_{s=0}$$

up to simple factors.

Anticyclotomic variation:

Let $\mathcal{O}_{p^s} = \mathbb{Z} + p^s \mathcal{O}_K$.

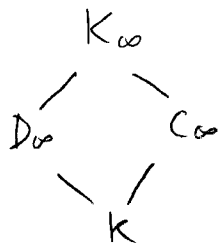
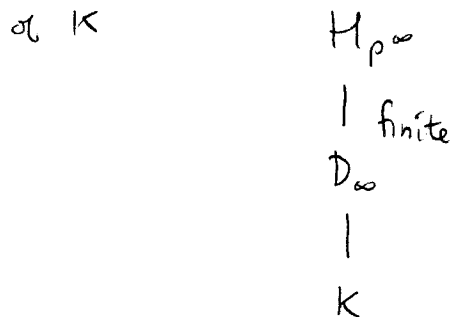
H_{p^s} = ring class field of \mathcal{O}_{p^s}

$$\text{Gal}(H_{p^s}/K) \cong \hat{K}^\times / K^\times \hat{\mathcal{O}}_{p^s}^\times$$

$$\cong \text{Pic}(\mathcal{O}_{p^s}).$$

Fact: $\text{Gal}(H_{p^s}/\mathbb{Q})$ is a generalized dihedral group.

Fact: $H_{p^\infty} = \bigcup H_{p^s}$ contains a unique \mathbb{Z}_p -extension



$$\begin{array}{ccc} \mathbb{C}/\mathcal{O}_{p^s} & \xrightarrow{h_s} & \mathbb{C}/(\mathcal{O}_{p^s} \cap \mathcal{N})^{-1} \in X_0(N)(H_{p^s}) \\ & & \downarrow \\ & & E(H_{p^s}) \\ & & \downarrow \\ & & h_s \in \text{Sel}(H_{p^s}, T_{\mathcal{O}_p} E) \end{array}$$

Let $\alpha \in \mathbb{Z}_p^\times$ be unit root of $x^2 - a_p(E)x + p$. Then

$$\frac{1}{\alpha^s} h_s - \frac{1}{\alpha^{s+1}} h_{s-1} \in \text{Sel}(H_{p^s}, T_{\mathcal{O}_p} E).$$

are norm compatible. Take the limit and then

Norm H_{p^s}/D_{∞} to get

$$Z_{\infty} \in S_{\infty} = \varprojlim \text{Sel}(D_s, T_{\mathcal{O}_p} E)$$

Set $\Lambda(D_{\infty}) = \mathbb{Z}_p \llbracket \text{Gal}(D_{\infty}/k) \rrbracket$. There exists

$\Lambda(D_{\infty})$ -adic height pairing

$$\langle , \rangle_{\infty} : S_{\infty} \times S_{\infty} \longrightarrow \Lambda(D_{\infty}).$$

Parisin-Risin constructs $\mathcal{L}(K_{\infty}) \in \Lambda(K_{\infty}) = \mathbb{Z}_p \llbracket \text{Gal}(K_{\infty}/k) \rrbracket$.

Identify $\Lambda(K_{\infty}) \cong \Lambda(D_{\infty}) \llbracket \text{Gal}(C_{\infty}/k) \rrbracket$

$$\cong \Lambda(D_\infty) \llbracket T \rrbracket$$

Expand $\mathcal{L}(K_\infty) = \mathcal{L}(D_\infty) + \mathcal{L}'(D_\infty)T + \mathcal{L}''(D_\infty)T^2 + \dots$

" 0 because of functional equation

Thm: (Howard): Assume p splits in K . Then

$$\langle Z_\infty, Z_\infty \rangle_\infty = \mathcal{L}'(D_\infty).$$

Hida Theory:

Let $\{f_\alpha\}$ be the Hida family containing the (p -stabilization of) the cusp form of E , i.e., there is a finite flat

$\mathbb{Z}_p \llbracket 1+p\mathbb{Z}_p \rrbracket$ -algebra R where each α is a "classical point",

$\alpha \in \text{Hom}_{\mathbb{Z}_p\text{-alg}}(R, \overline{\mathbb{Q}}_p)$, each $f_\alpha \in S_2(\Gamma_0(N) \cap \Gamma_1(p^2), \chi_\alpha, \overline{\mathbb{Q}})$.

Only consider f_α of even wt. For such f_α , χ_α has a square root. Let $f_\alpha^* = f_\alpha \otimes \chi_\alpha^{-1/2}$ trivial character.

Hida constructs rank 2 R -module with $G_{\mathbb{Q}}$ -action T

s.t. $T \otimes_{\alpha: R \rightarrow \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p = V(f_\alpha) = \text{Galois representation associated to } f_\alpha.$

$$\textcircled{2}: G_{\mathbb{Q}} \xrightarrow{\text{Ecycl}} \mathbb{Z}_p^\times \cong \mu_{p-1} \times (1+p\mathbb{Z}_p)$$

$$\downarrow$$

$$1+p\mathbb{Z}_p$$

$$\downarrow \gamma \mapsto \gamma^{1/2} \quad (\text{exists since } p \neq 2)$$

$$1+p\mathbb{Z}_p$$

$$\downarrow$$

$$\mathbb{Z}_p [1+p\mathbb{Z}_p]^\times$$

$$\downarrow$$

$$\mathbb{R}^\times$$

$$\text{Set } T^* = T \otimes \textcircled{2}^{-1}. \text{ Then } V(f_\alpha^*) = T^* \otimes_{\alpha: \mathbb{R} \rightarrow \bar{\mathbb{Q}}_p} \bar{\mathbb{Q}}_p.$$

For some $\alpha \in T_{\mathbb{Q}_p} E \otimes \bar{\mathbb{Q}}_p \cong V(f_\alpha^*)$, so $\exists T^* \rightarrow T_{\mathbb{Q}_p} E$.

Thm (Howard): There exists a ^{canonical} class $Z^{\text{bis}} \in \text{Sel}(K, T^*)$

$$\downarrow$$

$$Z \in \text{Sel}(K, T_{\mathbb{Q}_p} E).$$

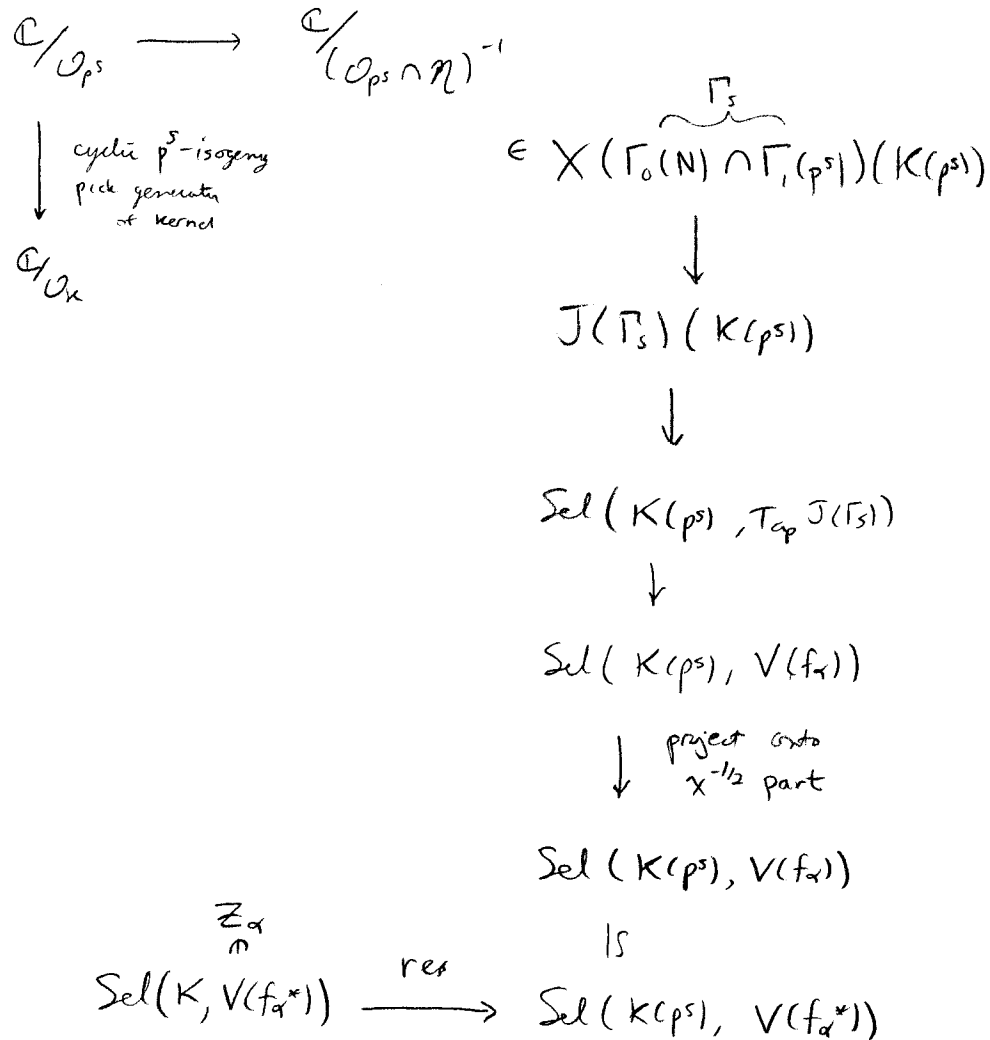
For any f_α^* , let Z_α be the image of Z^{bis} under

$$\text{Sel}(K, T^*) \rightarrow \text{Sel}(K, V(f_\alpha^*)).$$

Thm (Howard): Suppose f_α^* has weight 2. Then

$$Z_\alpha \neq 0 \iff \text{ord}_{s=1} L(f_\alpha^*/K, s) = 1.$$

What is Z_α ? Suppose $f_\alpha \in S_2(\Gamma_0(N) \cap \Gamma_1(p^s), \chi_\alpha, \bar{\mathbb{Q}})$.



Fact: The sign in the functional equation of f_α^* is constant with only finitely many exceptions.
 Let $w = \pm 1$ be the generic value.

Conjecture (Greenberg):

- 1) Suppose $w=1$. Then ← at center pt of punct. eq.
- Ⓐ $\text{ord } L(f_\alpha^*, s) = 0$ for a.e. f_α^*
 - Ⓑ $\dim \text{Sel}(\mathbb{Q}, V(f_\alpha^*)) = 0$ for a.e. f_α^* .
- 2) Suppose $w=-1$. Then

- Ⓐ $\text{ord } L(f_\alpha^*, s) = 1$ for a.e. f_α^*
- Ⓑ $\dim \text{Sel}(\mathbb{Q}, V(f_\alpha^*)) = 1$ for a.e. f_α^* .

Thm (Kato, Kitagawa, Mazur): Suppose $w=1$ and $\exists f_\beta^*$ s.t.
 $\text{ord } L(f_\beta^*, s) = 0$, then 1) Ⓐ and 1) Ⓑ hold.

Thm (Howard): Suppose $w=-1$. Suppose $\exists f_\beta^*$ of weight 2
s.t. $\text{ord } L(f_\beta^*, s) = 1$, then 2) Ⓑ holds.
and $\text{ord } L(f_\alpha^*, s) = 1$ for a.e. f_α^* of wt 2.

Sketch of proof:

- $\text{ord } L(f_\beta^*, s) = 1$
- \Rightarrow one can choose K s.t. $\text{ord } L(f_\beta^*/K, s) = 1$.
- $\Rightarrow Z_\beta \neq 0 \Rightarrow Z^{\text{bis}} \neq 0$.
- $\Rightarrow \text{Sel}(K, T^*) \cong \mathbb{R}$ by Euler systems
- $w=-1 \Rightarrow Z^{\text{bis}}$ fixed by $\text{Gal}(K/\mathbb{Q})$
- $\Rightarrow \text{Sel}(\mathbb{Q}, T^*) \cong \mathbb{R}$
- \Rightarrow Ⓑ.

$$Z^{\text{big}} \neq 0 \Rightarrow Z_\alpha \neq 0 \text{ for a.e. } f_\alpha^*$$

$$\Rightarrow \text{ord } L(f_\alpha^*/k, s) = 1 \text{ for a.e. } f_\alpha^* \text{ or } \text{ut } \omega.$$

$$\Rightarrow \text{ord } L(f_\alpha^*, s) = 1.$$

□