Construction of p-adic L-functions for unitary groups I

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Selmer groups, L-functions, and Galois deformations

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The long term goal is to relate p-adic L-functions to characteristic functions of Selmer groups.

First step: construct p-adic L-functions

There are conjectures on p-adic L-functions of motives (Coates, Perrin-Riou, Panchishkin)

Automorphic forms on Unitary groups

special (there are other kinds of groups)

general
- connection to GL_n
- these automorphic are generic as Hecke characters of CM fields are generic.

K = imaginary quadratic field

$c \in Gal(K/\mathbb{Q})$

W/K Hermitian space of dim. n. $u, v \in W$,

$\langle u, v \rangle$ linear in 1st variable and $\mathbb{C}$-linear in the second /
$\langle v, u \rangle = \overline{\langle u, v \rangle}$

$-W$ = space W and Hermitian form $-\langle , \rangle$, ie., $\langle , \rangle$ multiplied by -1.

$\partial W = W \oplus -W$ simplest kind of Hermitian space

$\langle (w,w'), (v,v') \rangle = 0 \quad \forall\, v, w \in W.$

totally isotropic subspace $\quad H = \mathcal{U}(\partial W)$

$P = \mathrm{Stab}_H(W^?) \quad C = \mathcal{U}(W)$

maximal parabolic subgroup $\quad P \to \begin{pmatrix} A & X \\ 0 & c({}^t\cancel{A}^{-1}) \end{pmatrix} = p \quad A \in GL_n(\mathcal{K})$

$A = A(p)$

$$m(A) = \begin{pmatrix} A & 0 \\ 0 & c\,{}^tA^{-1} \end{pmatrix} \in P \qquad m = \{ m(A) \}$$

R $\mathbb{Q}$-algebra

$$N(R) = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \in P \right\} \quad X \text{ runs over } \mathrm{Her}_n(R \otimes_{\mathbb{Q}} \mathcal{K}).$$

$$w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

I. Construction of the Eisenstein measure:

Doubling method for L functions of unitary groups. This is a generalization of Katz's construction for p-adic L-functions for Hecke characters.

Consider these subgroups of H with values in $\mathbb{Q}$, completions of $\mathbb{Q}$, and $\mathbb{A}$.

Let v be a completion of $\mathbb{Q}$.

$$\delta_v(p) = \left| N_{K_v/\mathbb{Q}_v} \circ \det A(p) \right|^{\frac{n}{2}} \quad (\text{modulus function})$$

$\delta_{\mathbb{A}}$ = adelic version $= |\ |\ \delta_v$

$\chi : K_{\mathbb{A}}^{\times}/K^{\times} \longrightarrow \mathbb{C}^{\times}$ Hecke character

χ defines an action of $M_{\mathbb{A}}$ by composition w/ det

$$\delta_{\mathbb{A}}(p,\chi,s) = \delta_{\mathbb{A}}(p)\,\chi\circ\det A(p)\,|N_{K/\mathbb{Q}}\circ\det(A(p))|_{\mathbb{A}}^{s}$$

$H(\mathbb{A}) = P(\mathbb{A})K_{\mathbb{A}}$ $K_{\mathbb{A}}$ some adelic maximal compact subgp.

$h\in H(\mathbb{A})$ $h = p(h)k(h)$ (not well-defined)

$$\sum_{\gamma\in P(\mathbb{Q})\backslash H(\mathbb{Q})} \delta_{\mathbb{A}}(p(\gamma h),\chi,s) = E(h,\chi,s)$$

$= K_{\mathbb{A}}$ invariant Eisenstein series.

This is a generalization of $\sum \frac{y^s}{|cz+d|^{2s}}$.

General Eisenstein series attached to functions

$$f(h,\chi,s) \in \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}\delta_{\mathbb{A}}(\cdot,\chi,s)$$

$$= \{ f : H(\mathbb{A})\to\mathbb{C} : f(ph) = \delta_{\mathbb{A}}(p,\chi,s)f(h)\}.$$

$E_f(h,\chi,s) = \sum_{\gamma\in P(\mathbb{Q})\backslash H(\mathbb{Q})} f(\gamma h,\chi,s)$ converges abs for s sufficiently large.

For an appropriate choice of f_∞, and s $f=\otimes f_v$, $f_v\in\mathrm{Ind}_{P(\mathbb{Q}_v)}^{H(\mathbb{Q}_v)}\delta_v(\cdot,\chi,s)$

E_f is holomorphic. (generalizes $\sum(cz+d)^{-k}$.)

$G = U(W) \simeq U(-W)$

$G \times G \hookrightarrow H$, $\quad G \times G \cap P = \Delta(G)$

$n=1$

$H = U(1,1) \supset SL_2 \qquad E_f|_{SL_2(\mathbb{R})}$ is essentially $y^{-k/2} \sum \frac{1}{(cz+d)^k}$

for appropriate s, k and $f_\infty(\gamma_y) = y^{-k/2}(c\cancel{z}+d)^{-k}$

π cuspidal automorphic representation of $G = U(W)$.

$\varphi \in \pi$ (could be a holomorphic mod. form on some Shimura variety).

$\varphi' \in \pi^\vee$

$\varphi'_\chi(g') = \varphi'(g')\chi^{-1} \circ \det(g') \qquad g' \in G = U(-W)$

$$Z(s, \varphi, \varphi', f, \chi) = \int_{G\times G(\mathbb{Q}) \backslash G\times G(\mathbb{A})} \underbrace{E_f}_{\text{has meromorphic cont.}}((g,g'),\chi,s)\, \varphi(g)\, \varphi'_\chi(g')\, dg\, dg'.$$

P.S. - Rallis proved ~~that~~ under certain hypotheses on f, φ, φ' (based on work of Garrett)

$$Z(s,\varphi,\varphi',f,\chi) = \prod_{v\in S} Z_v(s,\varphi,\varphi',f,\chi) \cdot \frac{L^S(s+\tfrac12, \pi, \chi, st)}{d_n^S(s,\chi)}$$

Z_v is an explicit local integral over $G(\mathbb{Q}_v)$.

$L^S(*, \pi, \chi, st)$ is a Langlands L-fctn. (with factors at S removed) $= \deg 2n / \mathbb{Q}$.

$n=1 \qquad \pi =$ Hecke char. α.

$L(s, \pi, \chi, St) = L(s, \alpha, \chi, st) = L(s, {}^{\alpha}/_{\alpha^c}\cdot\chi)$.

$n=2$: $\pi \nearrow$ holo. mod. form F
$\searrow$ Hecke char. α.

$$L(s,\pi, \cdot) = L(s, f_K \otimes \alpha/\alpha^c)$$

$$d_n(s,\chi) = \prod_{r=0}^{n-1} L(2s+n-r, \varepsilon_{K/\mathbb{Q}}^{n+1-r})$$

$S = \infty \cup S_f$. We can choose $f_v \in \mathrm{Ind}_{P_v}^{H_v}(-)$, $v \in S_f$, in such a way so that $Z_v(\text{—}) = \mathrm{vol}(K_v)$ where $K_v \subset G(\mathbb{Q}_v)$ is a sufficiently small compact open subgroup.

Z_∞ has been calculated up to rational factors by Garrett and in some cases exactly by Shimura/Garrett.

Thus,

$$Z(s,\varphi,\varphi',f,\chi) = \underset{\neq 0}{Z_\infty}\ \mathrm{vol}(K_S)\ \frac{L^S(s+1/2,\pi,\chi,St)}{d_n(s,\chi)}.$$

with $\mathrm{vol}(K_S) = \prod_{v\in S} K_v$.

If π, χ are arithmetic, then $L(s+1/2,\pi,\chi,St)$ is the L-function of a motive up to normalization.

Under favorable circumstances,

$$L^{mot}(s,\pi,\chi,st) = L(s, \rho_{\pi,\chi,\kappa})$$

$\uparrow$ shift by $\frac{n-1}{2}+\mu(\chi)$ $\qquad$ $\uparrow$ n-dim Gal. rep. of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$.

(M. Harris, Shimura)

<u>Thm:</u> Suppose s is a positive integer strictly to the right of the center of symmetry of $L^{mot}(s, —)$. Suppose s is critical in the sense of Deligne, then $P(s,\pi,\chi)^{-1} L^{mot}(s,\pi,\chi,st) \in \overline{\mathbb{Q}}$ and these values transform as expected under $Gal(\overline{\mathbb{Q}}/\mathcal{K})$.

where $P(s,\pi,\chi)^{-1}$ is essentially Deligne's period.

Now we would like to study the p-adic behavior of these algebraic numbers.

Harris, Li, Skinner deal only with varying χ. Ellen Eischen's thesis deals with the case of varying π and s.

$f_\infty \longleftrightarrow$ canonical automorphy factors

$$X(\partial W) = U(n,n)/U(n)\times U(n) \qquad (\text{for } n=1, \text{ this is } \mathfrak{h})$$

dim n^2.

$h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(n,n) \qquad J(h,\tau) = C\tau + D$

$\tau \in X(\partial W) \qquad J'(h,\tau) = C^{c\,t}\tau + D^c$.

$\mu, \kappa \in \mathbb{Z}$

$$\mathcal{J}_{\mu,\kappa}(h) = \det J(h,\tau_0)^{-\mu} \det J'(h,\tau_0)^{-\mu\kappa}$$

$T_0 =$ fixed point of $U(n) \times U(n)$

(Chosen w/ coordinates in $\mathcal{K}$)

$\mathbb{J}_{\mu,\varkappa}(h,s) \in \mathrm{Ind}_{P_\infty}^{H_\infty}(\varkappa, s) \quad = \mathbb{J}_{\mu,\varkappa} \, | \, \mathbb{J}_{\mu,\varkappa} |^{?}$ ← can't recall this

$f_\infty(h,s) = \mathbb{J}_{\mu,\varkappa}(h, \; s+\mu-\frac{n}{2}) =$

$f(h,s) = f_\infty(h,s) \; f_p \; f^{p,\infty}$ ← Simplest possible at $v \notin S$ f_v is fixed by max. compact. subgp.

$n(b) = \begin{pmatrix} 1_n & b \\ 0 & 1_n \end{pmatrix} \in U_P(\mathbb{A})$

$b \in \mathrm{Her}_n(\mathcal{K}_{\mathbb{A}})$

$\beta \in \mathrm{Her}_n(\mathcal{K}) \qquad \psi_\beta(n(b)) = \psi(\mathrm{Tr}_{\mathcal{K}_{\mathbb{A}}/\mathbb{A}} \circ \mathrm{Tr}(\beta b))$

$\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^\times$

$U_P(\mathbb{Q}) \backslash U_P(\mathbb{A}) \to \mathbb{C}^\times.$

$$E_{f_p,\beta}(h,\chi,s) = \int_{U_P(\mathbb{Q}) \backslash U_P(\mathbb{A})} E_{f_p}(u, h, \chi, s)\, \psi_\beta(u)\, du$$

f_p is the only variable. We choose it so that $E_{f_p} = 0$ if $\det \beta = 0$.

For $\det \beta \neq 0$. If $f^{\infty,p} = \prod_{v \nmid p\infty} f_v$ (factorizable)

then

$$E_{f_\beta}(h) = \underset{\substack{\uparrow \\ \text{constant}}}{c_{n,\varkappa}} \prod_v \int_{U_v} f_v(w u_v h_v, \chi_v, s)\, \psi_\beta(u_v)\, du_v.$$

$$= c_{n,\chi} \prod_v W_{\beta,v}(h_v, f_v, \chi_v, s).$$

Thm (Shimura): Suppose $\mu \neq 0$, $q \notin S$. $f_q(\cdot, \chi_q, s)$ invariant under K of $H(\mathbb{Z}_q)$, $f_q(1, \chi_q, s) = 1$. Then

$W_{\beta, q}(1, f_q, s) = 0$ unless β is Hermitian integral (almost)

In that case.

$$W_{\beta, q}(1, f_q, s) = \prod_{\eta=1}^{n} L_q(2s+\overset{?}{\cancel{q}}, \chi \varepsilon_K^{k-\eta})^{-1}$$

← this probably isn't correct, couldn't read the board!

$$E_{f_q}(\cdot, \chi, 0) = C_\infty^{-1} C_{n,k} \prod_{\eta=1}^{n} L(\eta, \chi \varepsilon_K^{k-\eta}) E_f(\cancel{\gamma}, \chi, 0)$$

↑ special value of Whit. fctn calc. by Shim.

holomorphic Eis series.

What's left has Fourier coeffs that depend only on f_p

(p splits in K, need to assume this throughout!)

Let $n = n_1 + \cdots + n_\ell$

$$Q = \begin{pmatrix} U(n_1) & & * \\ & \ddots & \\ 0 & & U(n_\ell) \end{pmatrix} \subset \underset{\displaystyle G(\mathbb{Q}_p)}{\underset{\wr\!\!\!\shortmid}{GL_n(\mathbb{Q}_p)}}$$

$$K_{Q_p} \subset GL_n(\mathbb{Z}_p)$$

$$= \{ k \in \mid k \equiv \begin{pmatrix} \text{⌐}_* \\ 0\text{⌐} \end{pmatrix} \bmod p \}$$

$$K = (Z_j) \quad j = 1, \dots, l$$

$Z_j \in GL_n(\mathbb{Z}_p)$

Z_{lj} have values in $\mathbb{Z}_p \quad l < j$

$p\mathbb{Z}_p \quad l > j.$

Let $\nu = (\nu_1, \dots, \nu_l)$ l-tuple of chars of $\mathbb{Z}_p^*$

$\phi_\nu : M_n(\mathbb{Q}_p) \to \mathbb{C}$

$$\phi_\nu(z) = \begin{cases} \prod \nu_n(\det Z_?) & Z \in K_{\mathbb{Q}} \\ 0 & o/w \end{cases}$$

I give up.. too fast and too hard to read...