

Hecke families for $GL(2)$ over totally
real fields

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Adelic groups, L -functions, and Galois
deformations

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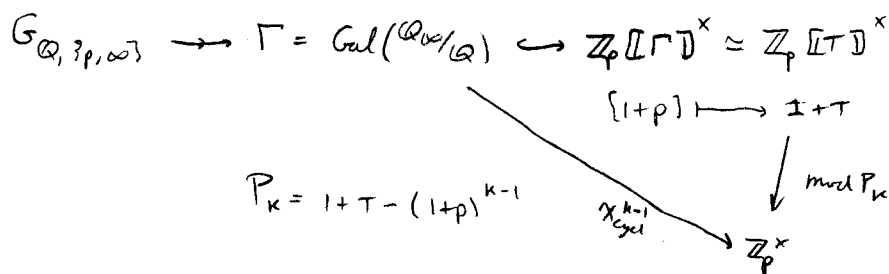
Let p be a prime.

There are lots of congruences mod powers of p between eigenforms $S_k(p^d; \mathbb{Z}_p)$.

GL(1) observation:

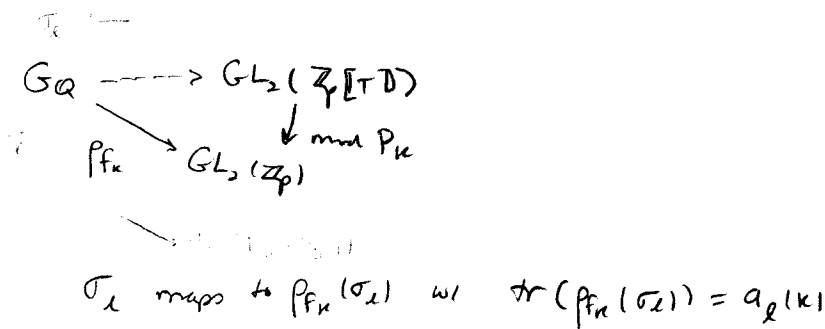
$n \equiv 1 \pmod{p}$ $\sum_{S \in \mathbb{Z}} r^S \mapsto r^n$ can be extended p -adically analytically.

$\mathbb{Q}_\infty/\mathbb{Q}$ cyclotomic \mathbb{Z}_p -extension

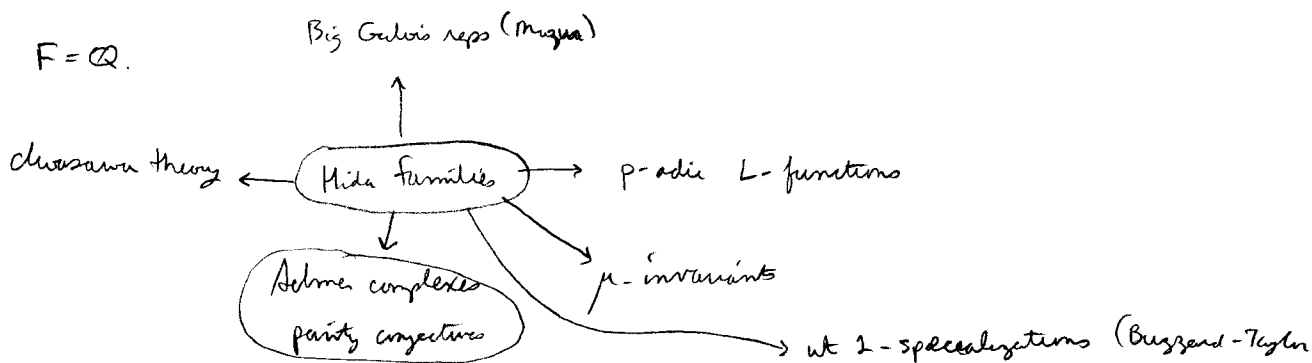


GL(2): $f_k = \sum_{n \geq 1} a_n(k) q^n$ $\sum_{n \geq 1} 1/n$

Want to extend $k \in \mathbb{Z}_{\geq 2} \mapsto a_n(k)$ to a p -adic analytic function



Assumption: ordinarity: $a_p(k)$ is a p -adic unit.



Constructions: p -adic modular forms $\bigcup_{k, \alpha} S_k(\Gamma_1(Np^\alpha), \mathbb{Z}_p)$.
 $\gcd(N, p) = 1$.

① Using (Betti) cohomology of Shimura varieties
 (Eichler-Shimura, Harder)

② Geometric definition $H^0(X_1(Np^\alpha)/\mathbb{Z}_p, \underline{\omega}^k)$ and use the
 cyclotomic tower. ↖ automorphic line bundle

II Modular curves ($F = \mathbb{Q}$):

← to avoid torsion.

$k=2$ Fix $N \geq 4$. tame level, we will drop it from the notation.

$\alpha \in \mathbb{N}$, $X_1(p^\alpha)$ = modular curve of level $\Gamma_1(Np^\alpha)$.

Hecke algebra $\mathcal{H}_\alpha = \mathbb{Z}_p[T_n : n \in \mathbb{N}] \subset H^1(X_1(p^\alpha), \mathcal{O}_p/\mathbb{Z}_p) = M_\alpha$, $\mathcal{H}_\alpha \subset \text{End}(M_\alpha)$

$X_{\alpha+1}$ gives by restriction $\mathcal{H}_{\alpha+1} \rightarrow \mathcal{H}_\alpha \subset M_\alpha \hookrightarrow M_{\alpha+1}$.
 \downarrow
 X_α

Set $M_\infty = \varinjlim M_\alpha \supseteq \varprojlim \mathcal{H}_\alpha = \mathcal{H}_\infty$.

* = Pontryagin dual: ($\mathbb{Z}_p \leftrightarrow \mathcal{O}_p/\mathbb{Z}_p$ coeff)

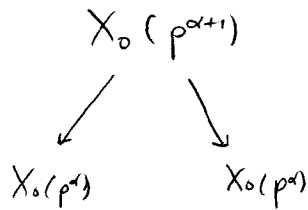
X_α
 \downarrow] étale of group $(\mathbb{Z}/p^\alpha)^\times \rightarrow \mathcal{H}_\alpha \subset \text{End}(M_\alpha)$.
 $X_0(p^\alpha)$

$\Rightarrow M_\infty^*$ is a $\mathbb{Z}_p[\varprojlim (\mathbb{Z}/p^\alpha)^\times] \cong \mathbb{Z}_p[\mathbb{Z}_p^\times]$ -module

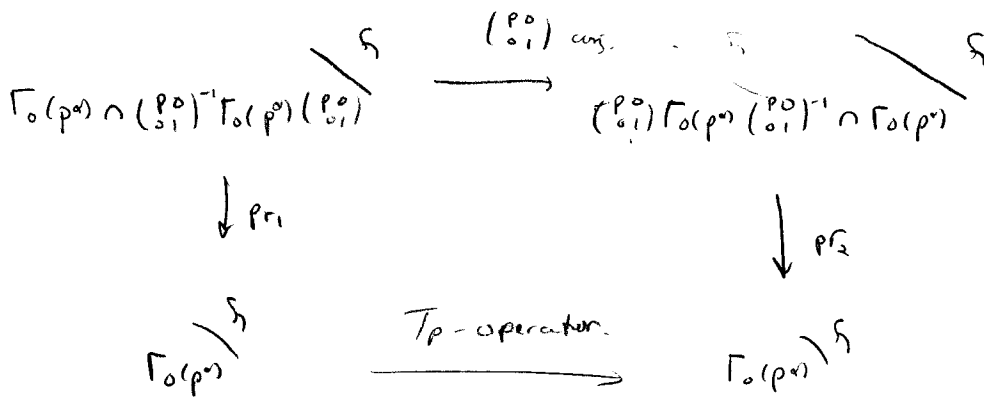
$M_{\infty}^{\text{ord}*} \subset M_{\infty}^*$ is the biggest direct \mathbb{Z}_p -factor where T_p is invertible.

$$h_{\infty} = h_{\infty}^{\text{ord}} \times h_{\infty}^{\text{ss}}$$

\mathbb{T}_p $U_p = \text{unit}$ $U_p = \text{top. nilpotent}$.



($N=1$ here)



So the ordinary assumption \Rightarrow

$$H_{\text{ord}}^i(X_0(p^{\alpha+1})) \cong H_{\text{ord}}^i(X_0(p^{\alpha})) \cong \dots \cong H_{\text{ord}}^i(X_0(p)).$$

\Rightarrow There are no ordinary families for Γ_0 .

It also shows that families for Γ_1 are the same as deforming central characters.

Thm (Hida): ① h_∞^{ord} is finite and flat over $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^*]]$, and

so also over $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$.

$M_\infty^{\text{ord},*}$ is free of finite type over Λ .

② $\alpha \geq 0$, $P_\alpha = (1+T)^{p^{\alpha-1}} - 1$, then

$$h_\infty^{\text{ord}} / P_\alpha h_\infty^{\text{ord}} \simeq h_\alpha^{\text{ord}}, \quad M_\infty^{\text{ord},*} / P_\alpha M_\infty^{\text{ord},*} \simeq M_\alpha^{\text{ord},*}.$$

III Hilbert modular case:

$F =$ totally real field, $[F:\mathbb{Q}] = d$, $\mathcal{I} = \text{Hom}_{\mathbb{Q}}(F; \mathbb{C})$, For $\mathbb{Q} \begin{matrix} \nearrow \mathbb{C} \\ \searrow \mathbb{Q}_p \end{matrix}$

B quaternion algebra over F (ex: $B = M_2(F)$). Ramified only possibly

at some infinite places $\mathcal{I}_B \subset \mathcal{I}$.

$$G = \text{Res}_{\mathbb{Q}}^F B^\times \quad (G = GL_2(F)/\mathbb{Q})$$

$K =$ open compact of $G(\mathbb{A}_f) = GL_2(\mathbb{A}_{F,f})$.

"

$$\prod K_v, \quad K_v \subset GL_2(\mathbb{Q}_{F,v}).$$

$$Y_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_\infty \rightarrow \text{max compact of } G(\mathbb{R}) \text{ center.}$$

$Y_K =$ Shimura variety of dim $\#(\mathcal{I} \setminus \mathcal{I}_B)^r$, quasiprojective, and smooth

if K is small enough.

Two cases: ① $r = 0$ or 1 , $r \equiv d \pmod{2}$

\swarrow Hida varieties
 \nwarrow Shimura curves

② $r=d$ $G = GL_2(F)$ (Hilbert modulus case).

Level at p : $K_0(p^\alpha) \supset K_1(p^\alpha) \supset K_{\text{aff}}(p^\alpha)$

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$	$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$
└──┘	└──┘	└──┘
No families	central character 1 + δ -variable family $\delta = \text{Leopoldt}$	local twist types d -variable

Weight: $w = (w_\tau, \tau + 1; w_0)$ $w_\tau \geq 0$ $w_\tau \equiv w_0 \pmod{2}$

$$v_\tau = \frac{w_\tau - w_0}{2}$$

$$M_\alpha^{(w)} = H^r(Y_H(p^\alpha), \mathcal{L}(w, E/O))$$

\hookrightarrow
 $h_\alpha^{(w)} \leftarrow \mathcal{O}[T(\mathbb{Z}/p^\alpha)/\mathcal{O}_F^*]$ \leftarrow local system $\otimes_{\mathbb{Z}} \text{Sym}^{w_\tau} \otimes \det \frac{w_0 - w_\tau}{2}$

$T = T_{\text{unr}} \subset B = B_{\text{ord}} \subset G$.

$$\begin{array}{c} Y_H(p^\alpha) \\ \downarrow \\ Y_0(p^\alpha) \end{array} \left[\begin{array}{c} T(\mathbb{Z}/p^\alpha)/\mathcal{O}_F^* \simeq (\mathcal{O}_F/p^\alpha)^2 / \mathcal{O}_F^* \\ (u, 0) \longleftarrow (u, z) \end{array} \right.$$

$$M_\infty^{(w)} = \varinjlim M_\alpha \quad h_\infty^{(w)} = \varprojlim h_\alpha^{(w)} \leftarrow \mathcal{O}[T(\mathbb{Z}_p)/\mathcal{O}_F^*] = \Lambda^{n,0}$$

U
 $M_\infty^{n,0}$ = nearly ordinary
 $h_\infty^{n,0} \times h_\infty'^{(w)}$

$$T_p = \prod_{v \neq p} T_v = T_p \times p^2$$

Thm (independence of wt. thdn):

$$H_{\infty}^{n,0}(w) \xrightarrow{\sim} H_{\infty}^{n,0}(0) = H_{\infty}^{n,0}$$

as $\Lambda^{n,0}$ -modules.

$$M_{\infty}^{n,0}(w) \xrightarrow{\sim} M_{\infty}^{n,0}(0) = M_{\infty}^{n,0}$$

where the action of $\Lambda^{n,0}$ on wt w is twisted by

$$\begin{aligned} \Lambda_n^{n,0} &\longrightarrow \Lambda_{n,1}^{n,0} \\ [u, z] &\longmapsto u^{\vee} z^{w_0} [u, z] \end{aligned}$$

idea behind proof:

$$\begin{aligned} M_{\infty}^{n,0}(w) &= \lim_{\alpha} H_{n,0}^r(Y_{11}(p^{\alpha}), \mathcal{Z}(w, E/0)) \\ &= \lim_{\alpha} H_{n,0}^r(Y_{11}(p^{\alpha}), \mathcal{Z}(w, P^{\text{rk}} \mathcal{O}/0)) \\ &\cong \lim_{\alpha} H_{n,0}^r(Y_{11}(p^{\alpha}), P^{\text{rk}} \mathcal{O}/0) \\ &\cong M_{\infty}^{n,0}. \end{aligned}$$