Hida families for GL(2) over totally real fields

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Selmer groups, L-functions, and Galois deformations

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Let p be a prime.

There are lots of congruences mod powers of p between eigenforms $S_k(p^\alpha; \mathbb{Z}_p)$.

GL(1) observation:

$n \equiv 1 \pmod{p}$ $\qquad \overset{\mathbb{Z}}{\underset{s}{\in}} \longmapsto n^s$ can be extended p-adic analytically.

$\mathbb{Q}_\infty / \mathbb{Q}$ cyclotomic $\mathbb{Z}_p$-extension

$$\begin{tikzcd} G_{\mathbb{Q}, \{p,\infty\}} \arrow[r] & \Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \arrow[r, hook] \arrow[drr, "\chi_{cycl}^{k-1}"'] & \mathbb{Z}_p[\![\Gamma]\!]^\times \simeq & \mathbb{Z}_p[\![T]\!]^\times \arrow[d, "\bmod P_k"] \\ & & & \mathbb{Z}_p^\times \end{tikzcd}$$

$[1+p] \longmapsto 1+T$

$P_k = 1 + T - (1+p)^{k-1}$

GL(2): $\qquad f_k = \sum_{n \geq 1} a_n(k) q^n$

Want to extend $k \in \mathbb{Z}_{\geq 2} \longmapsto a_n(k)$ to a p-adic analytic function

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, dashed] \arrow[dr, "\rho_{f_k}"'] & GL_2(\mathbb{Z}_p[\![T]\!]) \arrow[d, "\bmod P_k"] \\ & GL_2(\mathbb{Z}_p) \end{tikzcd}$$

σ_ℓ maps to $\rho_{f_k}(\sigma_\ell)$ w/ $\mathrm{tr}(\rho_{f_k}(\sigma_\ell)) = a_\ell(k)$

Assumption: ordinarity : $a_p(k)$ is a p-adic unit.

$F = \mathbb{Q}$.

$$\begin{tikzcd} & \text{Big Galois reps (Mazur)} & & \\ \text{Iwasawa theory} & \text{Hida families} \arrow[l] \arrow[u] \arrow[r] \arrow[d] \arrow[dr] \arrow[ddrr] & \text{p-adic L-functions} & \\ & \text{Selmer complexes, parity conjectures} & \text{μ-invariants} & \\ & & & \text{wt 1-specializations (Buzzard-Taylor} \end{tikzcd}$$

Constructions: p-adic modular forms $\bigcup_{k,\alpha} S_k(\Gamma_1(Np^\alpha), \mathbb{Z}_p)$.

$\gcd(N,p)=1$.

① Using (Betti) cohomology of Shimura varieties

(Eichler-Shimura, Harder)

② Geometric definition $H^0(X_1(Np^r)/\mathbb{Z}_p, \underline{\omega}^k)$ and use the Igusa tower.

$\underline{\omega}$ ← automorphic line bundle

II Modular curves $(F=\mathbb{Q})$:

$k=2$ Fix $N \geq 4$. (to avoid torsion) tame level, we will drop it from the notation.

$\alpha \in \mathbb{N}$, $X_1(p^\alpha)$ = modular curve of level $\Gamma_1(Np^\alpha)$.

Hecke algebra $h_\alpha = \mathbb{Z}_p[T_n : n \in \mathbb{N}] \curvearrowright H^1(X_1(p^\alpha), \mathbb{Q}_p/\mathbb{Z}_p) = M_\alpha$, $h_\alpha \subset \mathrm{End}(M_\alpha)$

$$\begin{array}{c} X_{\alpha+1} \\ \downarrow \\ X_\alpha \end{array} \quad \text{gives by restriction} \quad h_{\alpha+1} \twoheadrightarrow h_\alpha \;;\; M_\alpha \hookrightarrow M_{\alpha+1}.$$

Set $M_\infty = \varinjlim M_\alpha \curvearrowleft \varprojlim_\alpha h_\alpha = h_\infty$.

$* =$ Pontryagin dual: $(\mathbb{Z}_p \longleftrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ coeff$)$

$$\left.\begin{array}{c} X_\alpha \\ \downarrow \\ X_0(p^\alpha) \end{array}\right] \text{étale of group } (\mathbb{Z}/p^\alpha)^\times \longrightarrow h_\alpha \subset \mathrm{End}(M_\alpha).$$

$\Rightarrow$ M_∞^* is a $\mathbb{Z}_p[\![\varprojlim (\mathbb{Z}/p^\alpha)^\times]\!] \simeq \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$-module

$M_\infty^{\mathrm{ord}\,*} < M_\infty^{*}$ is the biggest direct $\mathbb{Z}_p$-factor where T_p is invertible.

$$h_\infty = h_\infty^{\mathrm{ord}} \times h_\infty^{ss.}$$

$\mathbb{T}_p$ $\quad U_p$ = unit $\quad U_p$ = top. nilpotent.

$$\begin{tikzcd} & X_0(p^{\alpha+1}) \arrow[dl] \arrow[dr] & \\ X_0(p^\alpha) & & X_0(p^\alpha) \end{tikzcd}$$

(N=1 here)

$$\begin{tikzcd} \Gamma_0(p^\alpha) \cap \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_0(p^\alpha) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \backslash \mathfrak{h} \arrow[r, "\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{conj.}"] \arrow[d, "pr_1"] & \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(p^\alpha) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cap \Gamma_0(p^\alpha) \backslash \mathfrak{h} \arrow[d, "pr_2"] \\ \Gamma_0(p^\alpha) \backslash \mathfrak{h} \arrow[r, "T_p\text{-operator}"] & \Gamma_0(p^\alpha) \backslash \mathfrak{h} \end{tikzcd}$$

So the ordinary assumption $\Rightarrow$

$$H^1_{\mathrm{ord}}(X_0(p^{\alpha+1})) \simeq H^1_{\mathrm{ord}}(X_0(p^\alpha)) \simeq \cdots \simeq H^1_{\mathrm{ord}}(X_0(p)).$$

$\Rightarrow$ There are no ordinary families for Γ_0.

It also shows that families for Γ_1 are the same as deforming central characters.

Thm (Hida): ① h_∞^{ord} is finite and flat over $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$, and

so also over $\Lambda = \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!] \simeq \mathbb{Z}_p[\![T]\!]$.

$M_\infty^{ord,*}$ is free of finite type over Λ.

② $\alpha \geq 0$, $P_\alpha = (1+T)^{p^{\alpha-1}} - 1$, then

$$h_\infty^{ord} / P_\alpha h_\infty^{ord} \simeq h_\alpha^{ord}, \qquad M_\infty^{ord,*} / P_\alpha M_\infty^{ord,*} \simeq M_\alpha^{ord,*}.$$

III Hilbert modular case:

F = totally real field, $[F:\mathbb{Q}] = d$, $I = \mathrm{Hom}_{\mathbb{Q}}(F; \overline{\mathbb{Q}})$, Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

B quaternion algebra over F (ex: $B = M_2(F)$). ramified only possibly at some infinite places $I_B \subset I$.

$G = \mathrm{Res}_{\mathbb{Q}}^F B^\times$ $\quad (G = GL_2(F)/\mathbb{Q})$

K = open compact of $G(\mathbb{A}_f) = GL_2(\mathbb{A}_{F,f})$.

$K = \prod K_v$, $K_v \subset GL_2(\mathcal{O}_{F,v})$.

$$Y_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_\infty$$

K_∞: max cmpct of $G(\mathbb{R})$ · center.

Y_K = Shimura variety of dim $\#(I \setminus I_B) = r$, quasiprojective, and smooth if K is small enough.

Two cases: ① $r = 0$ or 1, $r \equiv d \pmod 2$

$r = 0$: Hida varieties; $r = 1$: Shimura curves

② $r = d$ $\quad G = GL_2(F)$ (Hilbert modular case).

Level at p: $\quad K_0(p^\alpha) \supset K_1(p^\alpha) \supset K_{11}(p^\alpha)$

$$\underbrace{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}_{\text{No families}} \qquad \underbrace{\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}}_{\substack{\text{central} \\ \text{characters} \\ 1+\delta\text{-variable} \\ \text{families} \\ \delta = \text{Leopoldt}}} \qquad \underbrace{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}}_{\substack{\text{local twist types} \\ d\text{-variable}}}$$

Weight: $\quad w = (w_\tau,\ \tau \in I;\ w_0) \qquad w_\tau \geq 0 \qquad w_\tau \equiv w_0 \pmod 2$

$$v_\tau = \frac{w_\tau - w_0}{2}$$

$$M_\alpha^{(w)} = H^r(Y_H(p^\alpha), \mathcal{L}(w, E/\mathcal{O}))$$

$\circlearrowleft$ $\qquad \nwarrow$ local system $\bigotimes_\tau \mathrm{Sym}^{w_\tau} \otimes \det^{\frac{w_0 - w_\tau}{2}}$

$h_\alpha^{(w)} \leftarrow \mathcal{O}[T(\mathbb{Z}/p^\alpha)/\mathcal{O}_F^\times]$

$T = $ Torus $\subset B = $ Borel $\subset G$.

$$\left.\begin{array}{c} Y_{11}(p^\alpha) \\ \downarrow \\ Y_0(p^\alpha) \end{array}\right] \quad \begin{array}{l} T(\mathbb{Z}/p^\alpha)/\mathcal{O}_F^\times \simeq (\mathcal{O}_F/p^\alpha)^2/\mathcal{O}_F^\times \\ \begin{pmatrix} u & 0 \\ 0 & z \end{pmatrix} \longleftarrow (u, z) \end{array}$$

$$M_\infty^{(w)} = \varinjlim M_\alpha \qquad h_\infty^{(w)} = \varprojlim h_\alpha^{(w)} \longleftarrow \mathcal{O}[[T(\mathbb{Z}_p)/\overline{\mathcal{O}_F^\times}]] = \Lambda^{n.o}$$

$\cup\!\!\mid \qquad\qquad\qquad\qquad \shortparallel$

$M_\infty^{n.o}{}^{(w)}$ = nearly ordinary $\qquad h_\infty^{n.o\,(w)} \times h'^{(w)}_\infty$

$$T_p = \prod_{v \mid p} T_v = T_p \times p^?$$

Thm (Independence of wt. thm):

$$\mathfrak{h}^{n.o}_{\infty}(w) \xrightarrow{\sim} \mathfrak{h}^{n.o}_{\infty}(0) = \mathfrak{h}^{n.o}_{\infty}$$

as $\Lambda^{n.o}$-modules.

$$M^{n.o}_{\infty}(w) \xrightarrow{\sim} M^{n.o}_{\infty}(0) = M^{n.o}_{\infty}$$

where the action of $\Lambda^{n.o}$ on wt w is twisted by

$$\Lambda^{n.o}_{n} \longrightarrow \Lambda^{n.o}_{n}$$

$$[u,z] \longmapsto u^{v} z^{w_0} [u,z]$$

Idea behind proof:

$$M^{n.o}_{\infty}(w) = \varinjlim_{\alpha} H^{r}_{n.o}(Y_{11}(p^{\alpha}), \mathcal{L}(w, E/\mathcal{O}))$$

$$= \varinjlim_{\alpha} H^{r}_{n.o}(Y_{11}(p^{\alpha}), \mathcal{L}(w, p^{-\alpha}\mathcal{O}/\mathcal{O}))$$

$$\simeq \varinjlim_{\alpha} H^{r}_{n.o}(Y_{11}(p^{\alpha}), p^{-\alpha}\mathcal{O}/\mathcal{O})$$

$$\simeq M^{n.o}_{\infty}.$$