How many automorphic forms are there over imaginary quadratic fields?

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Selmer groups, L-functions, and Galois deformations

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to forms of cohomological type.

How many automorphic forms are there?

Restrict to forms of cohomological type.

How shall we count them? We count them as the level increases.

<u>Related:</u> understand the growth of cohomology of local systems in arithmetic quotients.

<u>Classical modular forms:</u>

$$Y(N) = \mathbb{H}^2 / \Gamma(N) \qquad \Gamma(N) \subset PSL_2(\mathbb{Z})$$

Eichler - Shimura: $H^1_p(Y(N), \mathrm{Sym}^{k-2}\, \mathbb{Q}^2) \qquad (k \geq 2)$

$k=2$: $\quad \dim_{\mathbb{Q}} H^1_p(Y(N), \mathbb{Q}) = \dim H^1(X(N), \mathbb{Q}) = 2 \text{ genus } X(N)$

$$\sim \text{const.} [\Gamma : \Gamma(N)]$$

$$\sim \text{const.} \operatorname{Vol}(Y(N)).$$

Similarly, $k \geq 2 \quad \dim_{\mathbb{Q}} H^1_p(Y(N), \mathrm{Sym}^{k-2}\, \mathbb{Q}^2) \sim \overset{\text{const.}}{c_k} \operatorname{Vol}(Y(N))$

If $k=1$, this is quite different.

Hilbert modular forms, and other classical cases are similar to this case.

Now consider $F =$ imag. quad. field, $G = GL_2/F$, $G_\infty = G(F \otimes_{\mathbb{Q}} \mathbb{R})$,

$K_\infty =$ max cmpct, Z center, $G_\infty / K_\infty Z = \mathbb{H}^3$.

A typical arithmetic quotient is $\quad Y = \mathbb{H}^3 / GL_2 \mathcal{O}_F$

$$Y(\mathfrak{n}) = \mathbb{H}^3 / (GL_2(\mathcal{O}_F)(\mathfrak{n})).$$

$H^1(Y(\mathfrak{n}), \mathbb{Q})$, $M = 3$-mfld , $\chi(M) = 0$. so no real info.

Euler char.

Let G_∞ be a semisimple real Lie group with unitary dual $\hat{G}_\infty$. Goal is to understand given $\pi \in \hat{G}_\infty$ the multiplicities with which such representations occur in the space of cusp forms for G_∞.

$G_\infty = \mathbb{G}(F \otimes_{\mathbb{Q}} \mathbb{R})$ $\mathbb{G}$ = some alg. gp.

K_∞ = max. compact

(means congruent to I_N mod q)

Choose $\mathbb{G} \hookrightarrow GL_N$. Define $G(q) = G_\infty \cap GL_N(q)$

Arithmetic lattice Γ, $\Gamma(q) = \Gamma \cap G(q)$. Define the arithmetic quotient

$$Y(q) = \Gamma(q) \backslash G_\infty / K_\infty .$$

Set $m(\pi, \Gamma(q))$ = mult. in which π occurs in reg. rep. of G_∞ on $L^2_{cusp}(\Gamma(q), G_\infty)$. Set $V(q) = \mathrm{vol}(Y(q))$.

some local system

Similarly, understand $H^i(Y(q), \mathcal{V})$ in terms of $V(q)$.

<u>Trivial remark:</u> $H^i(Y(q), \mathcal{V}) \ll V(q)$

$Y(1)$ - choose cell decomp., say M cells. Then $Y(q)$ has decomp with $[\Gamma : \Gamma(q)] \cdot M$ cells.

$$\Rightarrow \dim H^i(Y(q), \mathbb{Q}) << V(q).$$

The key issue is whether π is a discrete series or not. One expects a discrete series when has an auto. form coming from a Shimura variety. Everything else should not be a discrete series.

$$\lim_{N(q)\to\infty} \frac{m(\pi, \Gamma(q))}{V(q)} = \begin{cases} 0 & \pi \text{ not discrete series} \\ \neq 0 & \pi \text{ discrete series.} \end{cases}$$

$$\pi \to \text{non-tempered} \to m(\pi, \Gamma(q)) << V(q)^{1-\mu}, \quad \mu > 0$$

<u>Conjecture (Sarnak-Xue)</u>:

$$m(\pi, \Gamma(q)) << V(q)^{\frac{2}{p}+\varepsilon}$$

$$p = \inf\{p \geq 2 : \text{matrix coeff of } \pi \text{ are in } L^p\}$$

$$= \begin{cases} 2 & \pi \text{ discrete series} \\ 2 & \pi \text{ tempered} \\ >2 & \pi \text{ not tempered.} \end{cases}$$

For tempered (not discrete series) reps, $\quad m(\pi, \Gamma(q)) << \dfrac{\Gamma(q)}{\log V(q)}$

<u>Thm (C-Emerton)</u>: $\pi \in \hat{G}_\infty$. Suppose ~~either~~ G_∞ does not admit discrete series, then if q is prime in $\mathcal{O}_F$

$$m(\pi, \Gamma(q^k)) <<_{\Gamma,\pi,q} V(q^k)^{1-\frac{1}{\dim G_\infty}}.$$

Also,

$$\dim H^i(Y(q^k), \mathbb{Q}) << V(q^k)^{1-\frac{1}{\dim G_\infty}}$$

either if G_∞ has no discrete series or $i \neq \frac{1}{2}\dim G_\infty/K_\infty$.

Example: F = imag. quad. $q \subseteq \mathcal{O}_F$, $N(q) = p$ = prime

$$\dim H^1(\mathbb{H}^3/\Gamma(q^k), \mathbb{Q}) \ll p^{2k}$$

If $H^1(\mathbb{H}^3/\Gamma(q^k), \mathbb{Q}) \neq 0$, for any k, then $\geq p^k$.

Thm (C-Dunfield): There exist arithmetic lattices $\Gamma \subseteq GL_2\mathbb{C}$, q s.t. $\dim H^1(\mathbb{H}^3/\Gamma(q^k), \mathbb{Q}) = 0$ for all k.

Look at

$$\tilde{H}^i := \varprojlim_n \varinjlim_k H^i(Y(q^k), \mathbb{Z}/p^n\mathbb{Z})$$

If $G = \varprojlim \Gamma/\Gamma(q^k)$, then $\tilde{H}^i$ is a $\mathbb{Z}_p[[G]] = \Lambda$-module.

$\tilde{H}^i$ is cofinitely generated as a Λ-module.

Thm (Lazard): Λ is a Noetherian ring.

1. need to understand some elementary facts about co(-finitely) generated Λ-modules
2. control theorem: recover $H^i(Y(q), \mathbb{Q}_p)$ from $\tilde{H}^i$.
3. some input: exactly that $m(\pi, \Gamma(q^k)) = o(V(q^k))$.

Lemma (M. Harris): $G = \varprojlim \Gamma/\Gamma(q^k)$, $d = \dim G$

$$G_K = \ker(G \to \Gamma/\Gamma(q^k)).$$

If M is a cofinitely generated Λ-module, then

$$\dim(M^{G_K} \otimes \mathbb{Q}_p) = r(M)p^{dk} + O(p^{(d-1)k}).$$

Trivial consequence:

$$H^i(G_k, M) = O(p^{(d-1)k}) \quad \text{for } i \geq 1.$$

This is enough to deduce the theorem.

How big is $\tilde{H}^i$?

Construct homology analogs $\tilde{H}_i$.

Def: M f.g. Λ-module.

$$\text{codim}(M) = \inf\{i : \text{Ext}^i(M, \Lambda) \neq 0\}$$

Thm (C.-Emerton): If $G = GL_2/F$ = imag. quad, then

$$\text{codim}(\tilde{H}_1) = 1.$$

Go back to general G. Let q_0 = Borel-Wallach q_0, $d = \dim G$

$q_0 = \dfrac{d - l_0}{2}$ $\quad l_0$ = rank of G_∞ - rank max comp.

Conj: $\text{Ext}^{d-2q_0}(\tilde{H}_{q_0}, \Lambda) \text{ "=" } \tilde{H}_{q_0}.$