Families of Galois representations and families of automorphic forms -- an introduction

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Selmer groups, L-functions and Galois deformations

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Missed beginning of talk..

Lemma: Let $[E:\mathbb{Q}_p] < \infty$. Let $\rho : G_{F,S} \to GL_n(E)$ be continuous.

Then after conjugation, the image lies in $GL_n(\mathcal{O}_E)$.

Proof: Think of ρ as a v.s V over E of dim n w/ a cont. action of $G_{F,S}$.

Choose a $\mathcal{O}_E$-lattice $L' \subseteq V$. Take $L = \underbrace{G_{F,S} \cdot L'}_{\mathcal{O}_E\text{-module generated by}}$.

It is clear that $G_{F,S}$ preserves L. Continuity $\Rightarrow$ $L' \subseteq L \subseteq \frac{L'}{p^k}$

for some k $\Rightarrow$ L is a lattice. ∎

Given

$$\begin{tikzcd} \rho : G_{F,S} \arrow[r] \arrow[ddr, "\bar{\rho}"'] & GL_n(\mathcal{O}_E) \arrow[d] \\ & GL_n(\mathcal{O}_E/m_E) \arrow[d, phantom, "\parallel"] \\ & GL_n(\mathbb{F}_q) \end{tikzcd}$$

1. Understand all maps $\qquad\qquad\qquad$ ← hard!

$$\bar{\rho} : G_{F,S} \to GL_n(\mathbb{F}_q).$$

2. Given $\bar{\rho}$, understand all "lifts", i.e, all comm. diagrams

$$\begin{tikzcd} \rho : G_{F,S} \arrow[r] \arrow[dr, "\bar{\rho}"'] & GL_n(\mathcal{O}_E) \arrow[d] \\ & GL_n(\mathbb{F}_q) \end{tikzcd}$$

$\mathrm{Ker}\,(\,GL_n(\mathcal{O}_E) \to GL_n(\mathbb{F}_q)\,)$ is a (pro) solvable group.

(pro) p-group. Thus, we have methods from CFT to approach this problem.

Now start with $\bar{\rho} : G_{F,S} \to GL_n(k)$ $\quad k = \mathbb{F}_q$. We insist (mainly for technical reasons) that $\bar{\rho}$ is absolutely irreducible, i.e. if we consider $\bar{\rho}$ as going into $\bar{k}$ it is irreducible. The natural place to start is to try and lift to $GL_n(k[\varepsilon]/\varepsilon^2)$ (think of Hensel's lemma, construct the lift in steps).

Let $\mathcal{C}$ = category of complete local Noetherian rings $(A, \mathfrak{m})$ with a given isom. $A/\mathfrak{m} \xrightarrow{\sim} k$. Morphisms

$$\begin{array}{ccc} A & \to & A' \\ \downarrow & & \downarrow \\ A/\mathfrak{m} & = & A'/\mathfrak{m}' \end{array}$$

Think about maps

$$\rho : G_{F,S} \to GL_n(A).$$

Instead, think about a free rank-n A-module V_A with a cont. action of $G_{F,S}$.

<u>Def</u>: V_A is a <u>deformation</u> of V_k if $V_A \otimes_A A/\mathfrak{m} \simeq V_k$.

<u>Def</u>: V_A and V_A' are strictly equivalent if $\exists$ a comm. diag.

$$\begin{tikzcd} V_A \arrow[r, "\sim"] \arrow[d] & V_A' \arrow[d] \\ V_k \arrow[r, equal] & V_k \end{tikzcd}$$

$D: \mathcal{C} \longrightarrow \text{Sets}$. $D(A) =$ set of strict equivalence class of deformations V_A.

What is $D(k[\varepsilon]/\varepsilon^2)$?

If $A = k[\varepsilon]/\varepsilon^2$, we have an exact seq. of A-modules:

$$0 \to \underset{\varepsilon A}{\overset{\|}{k}} \longrightarrow A \longrightarrow k \to 0$$

$$\Downarrow$$

$$0 \to V_k \to V_A \to V_k \to 0$$

Given $c \in D(k[\varepsilon]/\varepsilon^2) \Rightarrow [c] \in \mathrm{Ext}^1_{k[G_{F,S}]}(V_k, V_k)$

Given $[c]$, choose a basis for V_k.

$$\sigma \in G_{F,S} \quad \begin{pmatrix} X(\sigma) & Y(\sigma) \\ 0 & X(\sigma) \end{pmatrix}$$

Fix a splitting $\psi^{-1}: V_k \to V_A$ (as v.s)

Given $\sigma \in G_{F,S}$, consider $\sigma\psi^{-1}(p) - \psi^{-1}(\sigma p)$

This gives a cocycle in $H^1(G_{F,S}, \underbrace{\mathrm{Hom}(V_k, V_k)}_{\mathrm{Ad}\, V_k})$

$$D(k[\varepsilon]/\varepsilon^2) = \mathrm{Ext}^1_{k[G_{F,S}]}(V_k, V_k) = H^1(G_{F,S}, \mathrm{Ad}\, V_k).$$

See $H^1(G_{F,S}, \mathrm{Ad}\, V_k)$ is finite: let $G_{F,S}$ act on $\mathrm{Ad}\, V_k$ via $\mathrm{Gal}(L/F)$

$$\begin{array}{ccccccc} 0 \to & H^1(\mathrm{Gal}(L/F), \mathrm{Ad}\, V_k) & \to & H^1(G_{F,S}, \mathrm{Ad}\, V) & \to & H^1(G_{K,S}, \mathrm{Ad}\, V) \\ & \| & & & & \| \\ & \text{finite} & & & & \mathrm{Hom}(G_{K,S}, \mathrm{Ad}\, V) \\ & & & & & \| \\ & & & & & \text{finite. (CFT).} \end{array}$$

<u>Thm (Mazur)</u>: The functor D is representable, i.e. there exist $R \in \mathcal{C}$ s.t. $D(A) = \mathrm{Hom}_{\mathcal{C}}(R, A)$.

This means, given any

$$\begin{tikzcd} G_{F,S} \arrow[r, "\rho"] \arrow[dr, "\bar\rho"'] & GL_n(A) \arrow[d] \\ & GL_n(k) \end{tikzcd}$$

$\exists$ a universal rep.

$$\rho^{univ} : G_{F,S} \longrightarrow GL_n(R)$$

s.t. we get ρ back via a map $\mathrm{Hom}_{\mathcal{C}}(R, A)$.

Let $S = \{p, \infty\}$, $F = \mathbb{Q}$, $n = 1$. p odd.

$$\begin{tikzcd} G_{\mathbb{Q},S} \arrow[rr] \arrow[dr] & & A^\times \\ & \mathbb{Z}_p^\times \arrow[ur] & \end{tikzcd}$$

Fix $\bar\rho : G_{\mathbb{Q},S} \to k^\times$. Then
trivial rep.

$$\begin{tikzcd} G_{\mathbb{Q},S} \arrow[rr, "\rho"] \arrow[dr] & & A^\times & 1 + m_A \\ & \mathbb{Z}_p^\times \arrow[ur] & 1+p\mathbb{Z}_p \arrow[ur] & \end{tikzcd}$$

$$\mathbb{Z}_p^\times = (\mathbb{Z}/p\mathbb{Z})^\times \oplus \mathbb{Z}_p, \qquad 1+p\mathbb{Z}_p = 1+p$$

$(\mathbb{Z}/p\mathbb{Z})^\times$ will go to 0.

Thus, $D(A) = 1 + m_A$.

$R = \mathbb{Z}_p[[T]]$.

$$\begin{tikzcd}[column sep=small] \mathrm{Hom}_e(& \mathbb{Z}_p[[T]], & A) = m_A \\ & \mathbb{F}_p \quad 0 & \mathbb{F}_p \arrow[from=1-2, to=2-2] \arrow[from=1-3, to=2-3] \end{tikzcd}$$

$$T \mapsto 0$$

Given R, what is $D(k[\varepsilon]/\varepsilon^2)$?

$$m_R \to \varepsilon k[\varepsilon]/\varepsilon^2$$

$$\begin{tikzcd} \mathrm{Hom}(R, & k[\varepsilon]/\varepsilon^2) \\ k & k \arrow[from=1-1, to=2-1] \arrow[from=1-2, to=2-2] \end{tikzcd}$$

$$\mathrm{Hom}(m, k) = \mathrm{Hom}(m/(m^2, p), k)$$

If $R = \mathbb{Z}_p[[T]]$, then $m/(m^2, p) = \mathbb{F}_p \Rightarrow D(\mathbb{F}_p[\varepsilon]/\varepsilon^2) = \mathbb{F}_p$.

$\mathrm{Ext}^1_{\mathbb{F}_p[G_{\mathbb{Q},S}]}(\mathbb{F}_p, \mathbb{F}_p)$ or $H^1(G_{\mathbb{Q},\{p,\infty\}}, \mathbb{F}_p)$

unique ext.
here..

Witt vectors

Let $d_1 = \dim H^1(G_{F,S}, \mathrm{Ad}\, V_k)$. $\Rightarrow R = W(k)[[T_1, \ldots, T_d]]/I$

What is the obstruction to lifting?

$$\begin{tikzcd} \rho : G_{F,S} \arrow[r] \arrow[dr, dashed] & GL_n(A/I) \\ & GL_n(A/I^2) \arrow[u] \end{tikzcd}$$

Form a set-theoretic lift γ:

$$c(\sigma,\tau) = \gamma(\sigma\tau)\gamma(\tau)^{-1}\gamma(\sigma)^{-1}$$

This measures the failure to be a hom.

$$= Id + \underbrace{M_n(I)}_{I \otimes Ad\, V_A/_I} \bmod I^2.$$

$$\in H^2(G_{F,S},\ Ad\, V_A/_I \otimes I)$$

If $H^2(G_{F,S}, Ad\, V_k) = 0$ then R = smth $\Rightarrow I = 0$

Let $d_2 = \dim H^2(G_{F,S}, Ad\, V_k)$, $I \otimes R/m$ has dim d_2.

# of gen of $I = d_2$

So we would like to know d_1 and d_2, but we can't say much about these.

However, the Euler char. formula gives $d_1 - d_2$.

Assume S contains all primes dividing p. Then

$$d_1 - d_2 = 1 + n^2[F:\mathbb{Q}] - \sum_{v|\infty} \dim H^0(G_v, Ad\, V_k)$$

n^2 if $G_v = \{1\}$

if $G_v = \{\pm 1\}$ ...

<u>$n=1$</u>:

$$d_1 - d_2 = 1 + r_1 + 2r_2 - r_1 - r_2 = 1 + r_2.$$

Conj: $d_1 - d_2$ = Krull dim R/pR. even $n=1$, correct Krull dim $\Leftrightarrow$ Leopoldt conj.

<u>$n=2$</u>:

$\bar\rho(c) = \pm\begin{pmatrix}1&0\\0&1\end{pmatrix}$ even $\quad$ $\bar\rho(c) = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ odd

$1+4-2=3$ $\quad$ $1+4-4=1.$

How does one try to prove $R = T$? (T anything here...)