

Families of Galois representations and families of  
automorphic forms -- an introduction

Frank Calegari

Selmer groups, L-functions, and Galois deformations

UCLA

3-24-2008

Lemma: Let  $[E:\mathbb{Q}_p] < \infty$ . Let  $\rho: G_{F,S} \rightarrow GL_n(\mathbb{Q}_E)$  be continuous.

Then after conjugation, the image lies in  $GL_n(\mathcal{O}_E)$ .

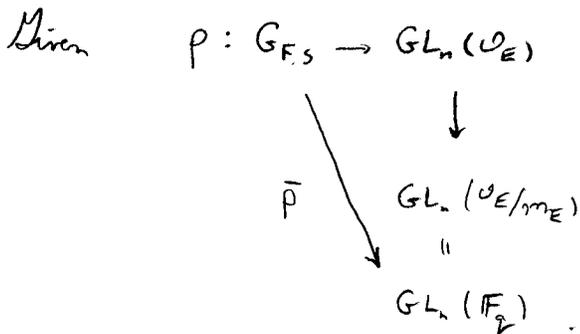
Proof: Think of  $\rho$  as a v.s  $V$  over  $E$  of dim  $n$  w/ a cont. action of  $G_{F,S}$ .

Choose a  $\mathcal{O}_E$ -lattice  $L' \subseteq V$ . Take  $L = \underbrace{G_{F,S} \cdot L'}_{\mathcal{O}_E\text{-module generated by}}$ .

$\mathcal{O}_E$ -module generated by

It is clear that  $G_{F,S}$  preserves  $L$ . Continuity  $\Rightarrow L' \subseteq L \subseteq \frac{L'}{p^k}$

for some  $k \Rightarrow L$  is a lattice.  $\square$

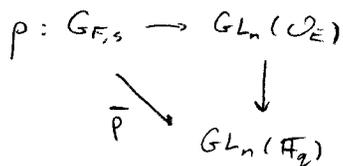


1. Understand all maps

← hard!

$$\bar{\rho}: G_{F,S} \rightarrow GL_n(\mathbb{F}_E).$$

2. Given  $\bar{\rho}$ , understand all "lifts", i.e., all comm. diagrams



$\text{Ker} (GL_n(\mathcal{O}_E) \rightarrow GL_n(\mathbb{F}_q))$  is a (pw) solvable group.

(pw) p-group. Thus, we have methods from CRT to approach this problem.

Now start with  $\bar{\rho}: G_{F,S} \rightarrow GL_n(K)$   $K = \mathbb{F}_q$ . We insist (mainly for technical reasons) that  $\bar{\rho}$  is absolutely irreducible, i.e., if we consider  $\bar{\rho}$  as going into  $\bar{K}$  it is irreducible. The natural place to start is to try and lift to  $GL_n(\mathbb{K}[[\epsilon]]/\epsilon^2)$  (think of Hensel's lemma, construct the lift in steps).

Let  $\mathcal{C}$  = category of complete local Noetherian rings  $(A, \mathfrak{m})$  with a given isom.  $A/\mathfrak{m} \cong K$ . Morphisms

$$\begin{array}{ccc} A & \rightarrow & A' \\ \downarrow & & \downarrow \\ A/\mathfrak{m} & = & A'/\mathfrak{m}' \end{array}$$

Think about maps

$$\rho: G_{F,S} \rightarrow GL_n(A)$$

instead, think about a free rank- $n$   $A$ -module  $V_A$  with a cont. action of  $G_{F,S}$ .

Def:  $V_A$  is a deformation of  $V_K$  if  $V_A \otimes_A A/\mathfrak{m} \cong V_K$ .

Def:  $V_A$  and  $V_{A'}$  are strictly equivalent if  $\exists$  a comm. diag.

$$\begin{array}{ccc} V_A & \xrightarrow{\sim} & V_{A'} \\ \downarrow & & \downarrow \\ V_{\mathfrak{k}} & = & V_{\mathfrak{k}} \end{array}$$

$D: \mathcal{C} \rightarrow \text{Sets}$      $D(A) =$  set of strict equivalence class of deformations  $V_A$ .

What is  $D(K[\epsilon]/\epsilon^2)$ ?

If  $A = K[\epsilon]/\epsilon^2$ , we have an exact seq. of  $A$ -modules:

$$0 \rightarrow \mathfrak{k} \rightarrow A \rightarrow \mathfrak{k} \rightarrow 0$$

"  $\epsilon A$

$\Downarrow$

$$0 \rightarrow V_{\mathfrak{k}} \rightarrow V_A \rightarrow V_{\mathfrak{k}} \rightarrow 0$$

Given  $\sigma \in D(K[\epsilon]/\epsilon^2) \Rightarrow \sigma \in \text{Ext}_{K[G_{F,S}]}^1(V_{\mathfrak{k}}, V_{\mathfrak{k}})$

Given  $\sigma$ , choose a basis for  $V_{\mathfrak{k}}$ .

$$\sigma \in G_{F,S} \quad \begin{pmatrix} X(\sigma) & Y(\sigma) \\ 0 & X(\sigma) \end{pmatrix}$$

Fix a splitting  $\psi^{-1}: V_{\mathfrak{k}} \rightarrow V_A$  (as v.s.)

Given  $\sigma \in G_{F,S}$ , consider  $\sigma \psi^{-1}(p) - \psi^{-1}(\sigma p)$

This gives a cocycle in  $H^1(G_{F,S}, \underbrace{\text{Hom}(V_{\mathfrak{k}}, V_{\mathfrak{k}})}_{\text{Ad } V_{\mathfrak{k}}})$

$$D(K[\epsilon]/\epsilon^2) = \text{Ext}_{K[G_{F,S}]}^1(V_{\mathfrak{k}}, V_{\mathfrak{k}}) = H^1(G_{F,S}, \text{Ad } V_{\mathfrak{k}}).$$



Thus,  $D(A) = 1 + m_A$ .

$$R = \mathbb{Z}_p \llbracket T \rrbracket.$$

$$\begin{array}{ccc} \text{Hom}_e(\mathbb{Z}_p \llbracket T \rrbracket, A) = m_A & & \\ \downarrow \quad \downarrow \quad \downarrow & & \\ \mathbb{F}_p & \mathbb{F}_p & \end{array}$$

Given  $R$ , what is  $D(K[\epsilon]/\epsilon^2)$ ?

$$\begin{array}{ccc} m_R \rightarrow \epsilon K[\epsilon]/\epsilon^2 & & \\ \text{Hom}(R, K[\epsilon]/\epsilon^2) & & \\ \downarrow \quad \downarrow & & \\ K & K & \end{array}$$

$$\text{Hom}(m, K) = \text{Hom}(m/m^2, K)$$

cf  $R = \mathbb{Z}_p \llbracket T \rrbracket$ , then  $m/m^2 = \mathbb{F}_p \Rightarrow D(\mathbb{F}_p[\epsilon]/\epsilon^2) = \mathbb{F}_p$ .

$$\text{Ext}_{\mathbb{F}_p[G_{q,r}]}^1(\mathbb{F}_p, \mathbb{F}_p) \text{ or } H^1(G_{q,r}, \mathbb{F}_p)$$

unique ext.  
here..

Let  $d_1 = \dim H^1(G_{F,S}, \text{Ad } V_K) \Rightarrow R = W(K) \llbracket T_1, \dots, T_{d_1} \rrbracket / I$  with vectors

What is the obstruction to lifting?

$$\begin{array}{ccc} \rho: G_{F,S} & \rightarrow & GL_n(A/I) \\ & \searrow & \downarrow \\ & & GL_n(A/I^2) \end{array}$$

Form a set-theoretic lift  $\gamma$ :

$$C(\sigma, \tau) = Y(\sigma\tau) Y(\tau)^{-1} Y(\sigma)^{-1}$$

This measures the failure to be a hom.

$$= \text{Id} + M_n(\mathbb{I}) \text{ mod } \mathbb{I}^2 \\ \underbrace{\quad\quad\quad}_{\mathbb{I} \otimes \text{Ad } V_A/\mathbb{I}}$$

$$\in H^2(G_{F,S}, \text{Ad } V_A/\mathbb{I} \otimes \mathbb{I})$$

If  $H^2(G_{F,S}, \text{Ad } V_A) = 0$  then  $R = \text{smooth} \Rightarrow \mathbb{I} = 0$ .

Let  $d_2 = \dim H^2(G_{F,S}, \text{Ad } V_A)$ ,  $\mathbb{I} \otimes R/\mathfrak{m}$  has  $\dim d_2$ .

# of gens of  $\mathbb{I} = d_2$

As we would like to know  $d_1$  and  $d_2$ , but we can't say much about these.

However, the Euler char. formula gives  $d_1 - d_2$ .

Assume  $S$  contains all primes dividing  $p$ . Then

$$d_1 - d_2 = 1 + n^2 [F:\mathbb{Q}] - \sum_{v|\infty} \dim H^0(G_v, \text{Ad } V_A)$$

$$H^2 \text{ if } G_v = \mathbb{Z}$$

$$\text{if } G_v = \mathbb{Z} \times \mathbb{Z}$$

$n=1$ :

$$d_1 - d_2 = 1 + r_1 + 2r_2 - r_1 - r_2 = 1 + r_2$$

Conj:  $d_1 - d_2 = \text{Kruddim } \mathcal{R}/\mathfrak{p}\mathcal{R}$ . even  $n=1$ , correct Kruddim  $\iff$  Leopoldt conj.

$n=2$ :

$$\bar{\rho}(c) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \bar{\rho}(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



How does one try to prove  $R=T$ ? (T. Anything here...)