$f \in S_2(\Gamma_0(N))^{new} \longleftrightarrow \pi$ on PGL_2

Given a Dirichlet character $\eta \longrightarrow \gamma(\eta) \in H_1(X_0(N), \mathbb{Q})$

$$(f, \gamma(\eta)) \leftrightsquigarrow L(f, \eta, 1).$$

If we want to study whether $L(f,\eta,1) \neq 0 \Leftrightarrow \gamma(\eta) \neq 0$ in $H_1(X_0(N), \bar{\mathbb{Q}})^{f\text{-component}}$

This appears to be hard to do in any kind of generality.

Example: Suppose we vary over quadratic characters. Is it true that $\exists\ \eta$ s.t. $\gamma(\eta) \neq 0$ in $H^1(X, \mathbb{Q})^f$? The answer is yes, but there is no direct proof with cycles $\gamma(\eta)$.

How do we really prove this?

$\Leftrightarrow \exists\ \eta$ s.t. $L(f,\eta,1) \neq 0$. There are three different looking proofs of this:

1. Waldspurger
2. Murty - Murty
3. Bump - Friedberg - Hoffstein

②: $L(f,\eta,s) = \sum \eta(n) a_n(f) n^{-s}$. The idea here is to average over η with $\underset{\displaystyle D}{\overset{\|}{\mathrm{cond}(\eta)}} < X$. (assume $\eta(-1) = -1$).

$$\frac{1}{X} \sum_{|D| \le X} \sum_{\eta} \eta(n) a_n(f) n^{-s} \leadsto \sum a_{m^2} m^{-2s}.$$

Show this is nonzero as $X \to \infty$ ~~at~~ $s = 2$.

At $s=1$ get $L(\mathrm{Sym}^2 f, 2) \neq 0$.

So at least something in this sum must be nonzero.

This does not tell how to find the particular η though.

③: Consider an Eisenstein series $\underset{E}{\vee}$ on a metaplectic group. E has Fourier coefficients involving $L(f, \eta, 1)$ and the constant term involves $L(\mathrm{Sym}^2 f, s) \neq 0$ at $s=2$. Thus, $E \neq 0$ at $s=2$. Look at the f.c. must be nonzero as well.

①: Here you don't see $\mathrm{Sym}^2 f$. He considers the Θ correspondence between PGL_2 and $\widetilde{SL}_2$. Fix an additive character ψ.

Local correspondence $\pi_v \xleftrightarrow{\Theta_\psi} \tilde{\pi}_v$

$\pi = \otimes \pi_v$ on PGL_2

$\downarrow$

$\tilde{\pi} = \otimes \tilde{\pi}_v$ abstract rep.

central critical value

Global theta lift $\pi \longrightarrow \Theta_\psi(\pi) = \begin{cases} 0 & L(\pi, \tfrac{1}{2}) = 0 \\ \tilde{\pi} & \text{o/w.} \end{cases}$

There is a reverse theta lift: Given $\tilde{\pi}$ on $\widetilde{SL}_2$ abstractly get $\pi = \otimes \pi_v$, $\pi_v = \Theta_\psi^{-1}(\tilde{\pi}_v)$.

Globally, $\tilde{\pi} \longrightarrow \Theta_{\bar{\psi}}(\tilde{\pi}) = 0$ iff $L(\pi, \tfrac{1}{2}) = 0$ and otherwise equals π.

Replace $\psi(x)$ by $x \longmapsto \psi(\xi x)$. Then

$$\pi \longrightarrow \tilde{\pi}$$

becomes

$$\pi \otimes \eta_\xi \longleftrightarrow \tilde{\pi}.$$

He also proves there exists some ψ for given $\tilde{\pi}$ s.t. $\Theta_\psi(\tilde{\pi}) \neq 0 \Rightarrow L(\pi, \tfrac{1}{2}) \neq 0$.

But, if we start with π on PGL_2, we don't know $\tilde{\pi}$ exists! The only way to find a candidate $\tilde{\pi}$

is by the trace formula. (Flicker).

Remark: None of these proofs construct the desired η.

Problem: You have to study $L(f,\eta,1)$ for all η at once.

Observation: To study $L(f,\eta,1)$, each is standard L-function for $\pi\otimes\eta$. $\pi\otimes\eta$ is a rep. of PGL_2 and all restrict to the same rep. on SL_2 (global L-packet) not irreducible.

Goal: Find a single invariant of the L-packet that should capture nontriviality of $L(f,\eta,1)$.

Remark: $Sym^2\otimes\det^{-1}$ is the first interesting rep. of PGL_2 = L-group of SL_2.

Toric Integrals: A key feature in ① is to consider

$$l(\varphi) = \int_{T_F\backslash T(A)} \varphi(t)\,dt = C_\varphi L(\pi,1/2) \qquad \varphi \text{ auto form in } \pi.$$

T = torus on PGL_2.

Focus on $\begin{pmatrix} t & \\ & 1\end{pmatrix}$ = diag. torus.

The functional $l:\pi\to\mathbb{C}$ is identically 0 iff $L(\pi,1/2)=0$.

Thm: Consider the SL_2 analogue of this:

$$l' : \varphi \longmapsto \int_{T'_F \backslash T'(\mathbb{A})} \varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} dt.$$

Then l' is not identically 0 on the packet of π.

$\iff \exists\, \eta$ s.t. $L(\pi, \eta, 1/2) \neq 0$.

Unwind thm: Notice that l' is a T'-invariant linear form on π. Look at local linear T'-invariant forms on SL_2.

Fix $\pi' = \otimes \pi'_v$ on SL_2.

Look at local T'-invariant form on π'_v.

Fact: The space of such forms has dim > 1 (usually)

dim $= 1, 2, 4$. Generic principal series has dim. 4.

Global space is huge!

Write down explicit forms: ~~l'~~ One candidate is the l' as above. Another way is to consider a Whittaker model for π'. Consider

$$\int_{T'} W_\varphi\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cancel{|t|^{s-1/2}\, dt}.$$

This can be computed explicitly and $L(\mathrm{Sym}^2 \otimes \det^{-1}, 2s)$ shows up. Analytically continue to get

$$\int W_\varphi\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} dt \sim L(\mathrm{Sym}^2 \otimes \det^{-1}, 1)$$

$$\neq 0.$$

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p5

<u>Plans for future work:</u> Prove directly that $l' \not\equiv 0$.

Deduce that $\exists\ \eta$ with $L(f,\eta,1)\neq 0$.

<u>Remarks:</u> There exist η for which $L(f,\eta,1)=0$ by the functional equation. This is visible on SL_2. $\begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$, $p \mid N$ switches elements in the L-packet.

To study l' look at the relative trace formula (Jacquet).

$f \in$ locally constant compactly support function on $SL_2\ /\ PGL_2$.

Consider the operator $\varphi \longmapsto R(f)\varphi$ on automorphic forms on G represented by a kernel $K_f(x,y)$. Want to integrate $K_f(x,y)$ on $T \times T$. Conj: on spec. side get

$$\sum_i \int \varphi_i(t) \int R(f)\varphi_j(t)$$

(φ_i = o.n. basis of L^2.)

gives a distribution on G. Want to show it detects functions in π - eigenspace of Hecke.

On the other side you get things like

$$\sum_\gamma \int_{t,s} f(t\gamma s)$$

this is for PGL_2, not exactly sure what get for SL_2.

over double cosets, γ is take over (?) $T_F \backslash G_F / T_F$

One should be able to see π in this distinguished term.