Algebraicity of L-functions for GL_2:

Goal: $f \in S_k(\Gamma_1(N), \chi)^{new}$ (k even)

There exist periods $\Omega_f^{\pm} \in \mathbb{C}^\times$ s.t. the following holds
if η is a primitive Dirichlet character of cond. M with $\eta(-1) = (-1)^{\pm}$
then

$$\frac{\tau(\bar{\eta})\, L(f, \eta, j+1)\, j!}{\Omega_f^{\pm}\, (2\pi i)^{j+1}} \in \overline{\mathbb{Q}}$$

with $\quad 0 \leq j \leq k-2$, and $\tau(\bar{\eta}) = \sum_{a=1}^{M} \bar{\eta}(a)\, e^{2\pi i a/M}$.

$f = \sum a_n q^n$

$L(f, \eta, s) = \sum_{n \geq 1} a_n \eta(n) n^{-s}$ for $Re(s)$ large. Note this does not converge in the range $1 \leq s \leq k-1$. So $L(f, \eta, j+1)$ is defined by analytic continuation.

Key formula:

$$\frac{\Gamma(s)}{(2\pi)^s} L(f, s) = \int_0^{i\infty} f(y)\, y^s \frac{dy}{y} \qquad \text{for } Re(s) \text{ large.}$$

$$\int_0^{\infty} \sum a_n e^{2\pi i n i y} (iy)^s \frac{dy}{y}$$

$$= \int_1^{\infty} + \int_0^1$$

converges for all s converges only for $Re(s) >> 0$.

For $Re(s) >> 0$ large integrate term by term:

$$\sum \int_0^\infty a_n e^{-2\pi n y} (iy)^s \frac{dy}{y}$$

$$= \sum a_n i^s \int_0^\infty e^{-2\pi n y} y^{s-1} dy$$

$= \ldots$ gives a nice formula for $Re(s)$ large.

One still must deal with $\int_0^1$ when $Re(s)$ is not necessarily large.

Key point: $\int_0^1 f(z)\,dz \leftrightarrow \int_1^\infty g(z)\,dz$ $\quad g(z) = f(-\frac{1}{z})$.

cusp form

converges for all s.

Main point: Get an expression for $\frac{\Gamma(s) L(f,s)}{(2\pi)^s}$ that is valid for all s via this trick. This gives a formula for $L(f,s)$ with s in the range of interest.

Interpretation: Replace the upper half-plane integral $\int_0^{i\infty}$, look at the integral on $\Gamma_1(N)\backslash\mathfrak{H}$ of $\omega_f = f(z)dz$.

(Assume $k=2$ for this part...)

$$\frac{L(f,1)}{2\pi} = \int_\gamma f(z)\,dz$$

where γ is the image in $\Gamma_1(N)\backslash\mathfrak{H}$ of path from 0 to $i\infty$. This integral actually converges. The integral $\int_\gamma \omega_f$ evaluates $L(f,s)$ at the point of interest.

For twisted L-function:

replace the path from $0 \rightsquigarrow i\infty$ by a sum of such paths:

$$\sum_{a=1}^{M} \left\{\frac{a}{M}, i\infty\right\} \eta(a) \rightsquigarrow \text{project to } \Gamma_1(N)\backslash\mathfrak{H} \text{ and integrate } \omega_f.$$

path from $\frac{a}{M} \rightsquigarrow i\infty$

<u>Summary:</u> For each η we get a path $\lambda(\eta)$ in the upper half plane as above, project to $\Gamma_1(N)\backslash\mathfrak{H}$ and get a compact path $\gamma(\eta)$ in the modular curve, integrate ω_f to get the L-value. To do higher weight the differential form must be modified.

<u>Remark:</u> In general, if a, b are elements in $\mathbb{P}^1(\mathbb{Q})$ one can make the same recipe with a path from $a \rightsquigarrow b$ in $\mathfrak{H}$. This is usually called the modular symbol $\{a,b\}$. One can then integrate ω_f on the image of this path in $X_1(N)$.

<u>Observe:</u> Given any differential form $\omega \in H^0(X_1(N), \Omega^1)_{\mathbb{C}}$, we can integrate $\int_{\gamma(\eta)} \omega$.

Get a functional on $H^0(X_1(N), \Omega^1)_{\mathbb{C}}$. This defines $\gamma(\eta)$ as an element of $H_1(X, \mathbb{R}) \otimes \mathbb{C}$.

elementary Hodge theory: each $\{a,b\} \to \gamma(a,b)$ defines an element of $H_1(X_1(N), \mathbb{R})$ ($\mathbb{R}$-dual of $H^0(X, \Omega^1)_{\mathbb{C}}$).

Theorem (Manin-Drinfeld): $\gamma(a,b)$ lies in $H_1(X_1(N), \mathbb{Q})$.

Assume this for now and show how this implies the main theorem ($k=2$).

Example: $N = 11$

$X_0(11)$ has genus 1 with two cusps $0, \infty$.

$H_1(X_0(11)) = \mathbb{Z} \oplus \mathbb{Z}$

$H_1(X_0(11), \mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$.

Pick this decomposition according to the $\pm$ eigenspaces of complex conjugation. Pick generators $\gamma^{\pm}$ of each eigenspaces.

$\{a, b\} = \{a,b\}^+ \oplus \{a,b\}^-$

$\{0, \infty\} = \{0,\infty\}^+$ (one can check this)

Integrate the unique diff. form ω_f on $\gamma^{\pm} \rightsquigarrow \Omega_f^{\pm} \in \mathbb{C}^{\times}$.

For any other element $\alpha \in H^1(X_0(N), \mathbb{Q})$ have

$\alpha = \alpha^+ \oplus \alpha^-$

$= c^+\gamma^+ \oplus c^-\gamma^-$

with $c^{\pm} \in \mathbb{Q}$.

Now $\displaystyle\int_{\alpha^{\pm}} \omega_f = c^{\pm} \int_{\gamma^{\pm}} \omega_f = c^{\pm}\Omega_f^{\pm}$. ← indep of α.

rational

Apply this to image of $\gamma(\eta) \in H_1(X_1(N), \overline{\mathbb{Q}})$ we get that $\displaystyle\int_{\gamma(\eta)} \omega_f = c^+(\eta)\Omega_f^+ + c^-(\eta)\Omega_f^-$

with $c^{\pm}(\eta) \in \overline{\mathbb{Q}}$.