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$X_0^n(p)$ Siegel variety of level p (drop the n sometimes)

V rep. of $GL(n)$

$\underline{\omega}_V$ } as last time

$\underline{\mathcal{H}}_W$ $\qquad \mathcal{H}_W = \mathcal{H}_V \qquad W = \mathrm{Ind}_{Q^-}^{Sp_{2n}} V|_Q$

Q Siegel parabolic

$V = st \rightsquigarrow \underline{\omega}_V = \underline{\omega}, \quad \mathcal{H}_V = \mathcal{H}^1_{dR}$

$\mu = (\mu_1, \dots, \mu_n), \quad \underline{\omega}_\mu := \underline{\omega}_{V_\mu}, \quad \underline{\mathcal{H}}_\mu := \underline{\mathcal{H}}_{V_\mu}$

$$\begin{array}{ccccc} H^0(X_0(p)_{/\mathbb{Z}_p}, \underline{\mathcal{H}}_\mu) & \longrightarrow & H^0(T_{\infty,\infty}, \mathcal{O}_{T_{\infty,\infty}}) & \xrightarrow{q\text{.exp}} & \mathbb{Z}_p[[q^{S_m}]] \\ \cup & & & & \\ H^0(X_0(p), \underline{\omega}_\mu) & & \longrightarrow & & \mathbb{Z}_p[[q^{S_m}]] \end{array}$$

<u>For $GL(2)$:</u> $\quad f \in H^0(X_0(p), \mathrm{Sym}^k \mathcal{H}^1_{dR}) \xrightarrow{C^\infty\text{-splitting}} f_\infty$, $\quad \xrightarrow{p\text{-adic splitting}} f_p$.

$K/\mathbb{Q}$ imag. quad., E/K CM

$\tau \in \mathfrak{h} \cap K$ $\qquad p$ splits in K. $\qquad$ know $f \neq 0 \Rightarrow f_\infty \neq 0$.

$$\frac{f_\infty(\tau)}{\Omega_\infty} = \frac{f_p(\tau_m)}{\Omega_p} \neq 0.$$

$W \qquad \mathrm{Fil}^0(W) = W^{N_Q} \subset W^{N_Q^2} \subset \dots$

$\mathrm{Fil}^q(\underline{\mathcal{H}}_V) \subset \underline{\mathcal{H}}_V$.

$\mathrm{Fil}^0(\underline{\mathcal{H}}_V) = \underline{\omega}_V$

$$\mathcal{N}_V^p(X_0(p)) = H^0(X_0(p), \mathrm{Fil}^p \mathcal{H}_V).$$

Hecke operators act on this space.

$$t = \mathrm{diag}(p^{e_1}, p^{e_2}, \ldots, p^{e_n}, p^{e_0-e_n}, p^{e_0-e_{n-1}}, \ldots, p^{e_0-e_1})$$

$$e_1 \le e_2 \le \cdots \le e_n \le e_0.$$

$$U_t = ItI \circlearrowright \mathcal{N}_{V_\mu}(X_0(p))$$

We choose the following normalization.

$$\frac{\mu(p^{e_1}, \ldots, p^{e_n})^{-1}}{[S_n(\mathbb{Z}_p) : tS_n(\mathbb{Z}_p)t^{-1}]} \times \text{usual action}$$

Families: $\mathcal{X} = \mathrm{Hom}_{cts}(T(\mathbb{Z}_p), \bar{\mathbb{Q}}_p^\times)$ $\qquad T(\mathbb{Z}_p) = (\mathbb{Z}_p^\times)^n$

μ alg wt $\in \mathcal{X}$.

$\mathcal{U} \subseteq \mathcal{X}$ affinoid.

$\mathcal{O}(\mathcal{U})$ = rigid analytic functions on $\mathcal{U}$.

slope: Θ = character of Hecke algebra taking values in $\bar{\mathbb{Q}}_p^\times$.

Θ is finite slope if $\Theta(U_t) \neq 0 \quad \forall t$.

$v_p(\Theta(U_t)) = e_0 s_0 + e_1 s_1 + \cdots + e_n s_n.$

$\underline{s} = (s_0, \ldots, s_n) \in \mathbb{Q}^n$

s = slope = $\max(s_i)$.

V alg. rep. of $GL(n)$

$$F \in \mathcal{N}_{V,\mathcal{U}}^{\le s} \subset V \otimes \mathcal{O}(\mathcal{U})[[q^{S_n}]].$$

$$\mathcal{V}_{V,\mu} : H^0(X_0(p)/_{\mathbb{Z}_p}, \underline{\mathcal{H}}_{V\otimes V_\mu}) \hookrightarrow H^0(T_{\infty,\infty}, V \otimes \mathcal{O}_{T_{\infty,\infty}}) \hookrightarrow V \otimes \mathbb{Z}_p[\![q^{S_n}]\!].$$

Such a F must satisfy evaluation of $\mu \in \mathcal{U}(\overline{\mathbb{Q}}_p)$ is $\mathcal{V}_{V,\mu}(f)$ with $f \in \mathcal{N}^{\leq p}_{V\otimes V_\mu}$ and with f of slope $\leq s$.

Gauss-Manin criterion:

$$\mathcal{H} \xrightarrow{\nabla} \mathcal{H} \otimes \Omega_X$$

For any

$$V \longrightarrow \mathcal{H}_V \longrightarrow \underline{\mathcal{H}}_V \otimes \Omega.$$

$V = St \otimes \cdots \otimes St$ can deduce this via Leibnitz rule.

Kodaira-Spencer map:

$$\begin{tikzcd} \underline{\omega} \subset \mathcal{H} \arrow[r] \arrow[dr] & \mathcal{H} \otimes \Omega^1 \arrow[d] \\ & \omega^\vee \otimes \Omega^1 \end{tikzcd}$$

$$\begin{tikzcd} \omega \otimes \omega \arrow[r] \arrow[d] & \Omega^1_X \\ \mathrm{Sym}^2 \omega \arrow[ur, "\simeq"'] & \end{tikzcd}$$

$$\begin{tikzcd} \nabla : \mathcal{H}_V \arrow[r] \arrow[dr, "\nabla"'] & \mathcal{H}_V \otimes \mathrm{Sym}^2(\underline{\omega}) \\ & \underline{\mathcal{H}_V} \otimes \underline{\omega}_{\mathrm{Sym}^2(St)} \\ & \mathcal{H}_{V \otimes \mathrm{Sym}^2(St)} \arrow[u, leftrightarrow] \end{tikzcd}$$

Kodaira-Spencer for $\hat{\mathcal{G}}/\mathbb{Z}[[q^{S_n}]]$ $\quad (\underline{\omega}_i)$: can. basis

$$\mathrm{Sym}^2\,\underline{\omega} \xrightarrow{\sim} \Omega^1_{\mathbb{Z}[[q^{S_n}]]} \qquad q_{ij} \quad \begin{matrix} 1/2 & ij & ji & i\neq j \\ 1 & ii \end{matrix}$$

$$\omega_i \otimes \omega_j \longleftrightarrow \frac{dq_{ij}}{p_{ij}}$$

Using the unit root splitting over the ordinary part

$$\underline{\omega}_V \xrightarrow{\nabla^{p\text{-adic}}} \underline{\omega}_{(V\otimes \mathrm{Sym}^2 \mathrm{St})}$$

$$\nabla(\omega_i) \in \mathcal{U}.$$

$$F \in H^0(X_0(p)^{-}, \underline{\mathcal{H}}_V)$$

$$(F)_{q\text{-exp}} = \sum_T c(T)\, q^T \qquad c(T) \in V.$$

$$(\nabla F)_{q\text{-exp}} = \sum_{\substack{T \\ i,j}} T_{i,j}\, e_i \otimes e_j \otimes c(T)\, q^T$$

$(e_i \otimes e_j)$ canonical basis of $\mathrm{Sym}^2(\mathrm{St})$.

$$\nabla^q : \mathcal{H}_V \longrightarrow \mathcal{H}_{V\otimes \mathrm{Sym}^q(\mathrm{Sym}^2(\mathrm{St}))}$$

$$Sp_{4n} \supset Sp_{2n} \times Sp_{2n}$$

$$F \in H^0(X_0^{2n}(p), \underline{\mathcal{H}}_{\det^{k_0}}) \qquad \text{some } k_0 \geq 2.$$

$$\left(\nabla^q F\big|_{X_0^n(p)\times X_0^n(p)}\right)_{q\text{-exp}}$$

$$\mathrm{Sym}^2(St_{2n})\Big|_{GL(n)\times GL(n)} = \mathrm{Sym}^2(St_n)\boxtimes \mathrm{triv}$$
$$\oplus\ \mathrm{triv}\boxtimes \mathrm{Sym}^2(St_n)$$
$$\oplus\ St_n \boxtimes St_n.$$

$$\left(\begin{array}{c|c} A & B \\ \hline {}^tB & D \end{array}\right)$$

$$\mathrm{Sym}(\mathrm{Sym}^2(St_{2n}))\Big|_{GL(n)\times GL(n)}$$

For any $V \subset \mathrm{Sym}(\mathrm{Sym}^2(St_{2n}))$. you can find

$$V \boxtimes V \otimes V_\mu \boxtimes V_\mu$$
$$= (V \otimes V_\mu) \boxtimes (V \otimes V_\mu).$$

$St \boxtimes St \simeq M_n \quad (x_{ij})$

$\mathrm{Sym}(St \boxtimes St)$ = polys. in x_{ij}

$\det(x_{ij}) \longleftrightarrow \det \boxtimes \det$

minors of size $q \times q$ $\longleftrightarrow$ miniscule rep. $\Lambda^q St_n$

$1 \le q \le n$ $\longleftrightarrow (\underbrace{1,\dots,1}_{q},0,\dots,0)$

$$\det(x_{ij})_{\substack{1\le i\le q\\ 1\le j\le q}}.$$

$\mu = (\mu_1,\dots,\mu_n)$

$= (\mu_1-\mu_2)(1,0,\dots,0) + (\mu_2-\mu_3)(1,1,0,\dots,0) + \dots + \mu_n(1,\dots,1).$

$$\mathrm{Sym}^{(\mu_1-\mu_2)}(St)\otimes \mathrm{Sym}^{(\mu_2-\mu_3)}\Lambda^2 St \otimes\dots\otimes (\Lambda^n St_n)^{\mu_n} \twoheadrightarrow V_\mu.$$

$x_{11}^{\mu_1-\mu_2}\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{vmatrix}^{\mu_2-\mu_3} \cdots \det(x_{ij})^{\mu_n}$ highest weight vector

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One considers

$$\mathrm{Sym}(\mathrm{Sym}^2(St_{2n}))\Big|_{GL_n\times GL_n} \longrightarrow (V\otimes V_\mu)\boxtimes(V\otimes V_\mu).$$

$$\nabla_{V,\mu} : \underline{\mathcal{H}}_{\det^{k}} \longrightarrow \underline{\mathcal{H}}_{\det\otimes \mathrm{Sym}^q(\mathrm{Sym}^2 st)} \xrightarrow[\text{and then proj}]{\text{rest. to } X_0^n(p)\times X_0^n(p)} \mathcal{H}^{X_0^n(p)}_{V\otimes V_\mu} \boxtimes \mathcal{H}^{X_0^n(p)}_{V\otimes V_\mu}.$$

$$F=\sum_T c(T)q^T \in \mathbb{Z}_p[\![q^{S_n}]\!]$$

$$V_{V,\mu}\left(\nabla_{V,\mu,k}F\right)_{q\text{-exp}} = \sum_{T=\begin{pmatrix} A & B\\ {}^tB & D\end{pmatrix}} P_V(A,D)\, b_{11}^{\mu_1-\mu_2} \begin{vmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{vmatrix}^{\mu_2-\mu_3} \cdots \det B^{\mu_n}\, q^A\otimes q^D$$

$B=(b_{ij})$

$P_V(A,D)=$ polynomial in A and D taking values in V.

Assumption: $C_T(F)=0$ if $\exists\ 1\le q\le n$ s.t.

$$p \;\Big|\; \begin{vmatrix} b_{11} & \cdots & b_{q1}\\ b_{1q} & \cdots & b_{qq}\end{vmatrix}.$$

$E(\phi,s)$ $\quad \phi\in \mathrm{Ind}_Q^{Sp_n}\chi.$

① $\nabla E(\phi,s)$ is $E(\phi',s)$ where ϕ' only changes the infinite sections.

② We may choose the section ϕ_p so that this assumption is satisfied.

Consequence: $\exists\ \Phi_{V,F} \in (V\otimes\Lambda[\![q^{S_n}]\!])\boxtimes(V\otimes\Lambda[\![q^{S_n}]\!])$

$(\Lambda=\Lambda_{\mathfrak{X}} \cong \mathbb{Z}_p[\Delta][\![T_1,\dots,T_n]\!])$ s.t.

$(\text{evaluation at } \mu) \otimes (\text{evaluation at } \mu)\, (\Phi_{V,F}) = q\text{-exp}(D_{V,\mu} F)$.

Next step is to construct and apply a finite slope projector.