$K \subseteq GSp_{2n}(\mathbb{A}_f)$, $K \supset K_1(N) = \{ g \in \underset{\shortparallel}{\overset{G}{}} GSp_{2n}(\hat{\mathbb{Z}}) : g \bmod N \in \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \}$ (maximal at p)

$(n \geq 2)$ (to avoid dealing w/ compactification)

$X = X_K$ Siegel modular variety of level K. It classifies

$(A, \lambda, \alpha)_{/S}$, A/S abelian scheme$/S$, S a $\mathbb{Z}[\frac{1}{N}]$-scheme

$\lambda : A \longrightarrow {}^tA$ principal polarization

$\alpha : T_\ell(A) \simeq \mathbb{Z}_\ell^{2n}$ (mod K)

$\langle , \rangle$ $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Scheme defined over $\mathbb{Z}[\frac{1}{N}]$

$$X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K Z_\infty = \coprod_i \Gamma_i \backslash \mathfrak{H}$$

$$\begin{array}{c} \mathcal{A} \\ \downarrow \pi \\ X \end{array}$$

$\underline{\omega} = \pi_* \Omega^1_{A/X}$ locally free sheaf of rank n

$\underline{\mathcal{H}} = R^1 \pi_* \Omega^\bullet_{A/X}$ locally free sheaf of rank $2n$

$$0 \longrightarrow \underline{\omega} \longrightarrow \underline{\mathcal{H}} \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow 0$$

$$\begin{array}{c} H^1(A,\mathcal{O}_A) \\ \wr\wr \\ \mathrm{Lie}({}^tA) = \underline{\omega}^\vee \text{ by } \lambda. \end{array}$$

$\langle , \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{O}_X$ Poincaré duality.

$(\underline{\mathcal{H}}, \langle , \rangle) \cong (\mathcal{O}_X^{2n}, J)$ locally.

(ρ, V) an algebraic rep. of $GL_n \xrightarrow{\rho} GL(V)$

$\underline{\omega}_V$

$\mathcal{T}_{\underline{\omega}} = \underline{\mathrm{Isom}}_X(\mathcal{O}_X^n, \underline{\omega})$

Urban

6-16-10

pg 2

$\underline{\omega}_V = V \times^{GL_n} \mathcal{T}_{\underline{\omega}}$ $\qquad (v,\phi) \qquad (\rho(g)v, \phi) \sim (v, \phi\circ g)$

$V = St \rightsquigarrow \qquad (v,\phi) \longmapsto \phi(v)$

$$\underline{\omega}_{St} \xrightarrow{\sim} \underline{\omega}.$$

Example: $V = \det^{k} \qquad \underline{\omega}_V = (\Lambda^n \underline{\omega})^{\otimes k}$

$\underline{\omega} \subset \underline{\mathcal{H}} \xleftarrow{\sim} \mathcal{O}_X^n \subset \mathcal{O}_X^{2n}$ $\qquad Q$ parabolic stabilizing

$\langle , \rangle \qquad\qquad \langle , \rangle_J$ $\qquad$ this flag. $\subset Sp_{2n}$

$$Q = \begin{pmatrix} g & * \\ 0 & {}^t g^{-1} \end{pmatrix}$$

$$\mathcal{T}_{\underline{\mathcal{H}}} = \underline{\mathrm{Isom}}_X \left(\mathcal{O}_X^n \subset \mathcal{O}_X^{2n}{}_J,\ \underline{\omega} \subset \mathcal{H}_{\langle,\rangle} \right)$$

(ρ_W, W) rep. of Q. fiber product over Q

$$\underline{\mathcal{H}}_W := W \times^Q \mathcal{T}_{\underline{\mathcal{H}}}$$

if $W = \mathrm{Res}_Q (St_{Sp_{2n}}) \longrightarrow \underline{\mathcal{H}}_W = \underline{\mathcal{H}}$

Example: $n=1 \qquad W = \mathrm{Sym}^k(\mathrm{Res}_Q\ St_{GL(2)})$

$$\underline{\mathcal{H}}_W = \mathrm{Sym}^k(\mathcal{H}^1_{dR})$$

$H^0(X, \underline{\mathcal{H}}_W)$, $\quad H^0(X, \underline{\omega}_V)$

$\overset{\psi}{f}$ is a rule $(A, \lambda, \alpha)/\mathrm{Spec}(R)$, (ω_i) basis of $\underline{\omega}_{A/R}$

$f(A,\lambda,\alpha,\omega_i) \in V_R = V_{\mathbb{Q}} \otimes R$, $g \in GL_n(R)$, $\omega_i' = g \cdot \omega_i$

Then

$$f(A,\lambda,\alpha,\omega_i') = \rho_V(g)^{-1} f(A,\lambda,\alpha,\omega_i).$$

q-expansion: Chai-Faltings defined toroidal compactifications $\overline{X}_K$

$\mathcal{G}$ at a zero dimensional cusp

$\downarrow$ $\quad \mathcal{G}/\mathbb{Z}[\![q^{S_n}]\!]$

$\overline{X}$ $\quad S_n$ = semi-group of $\frac{1}{2}$ integral $n \times n$ symmetric matrices, ≥ 0.

$$\hat{\mathcal{G}}/_{\mathbb{Z}[\![q^{S_n}]\!]} \cong \hat{\mathbb{G}}_m^n$$

$$(\omega_{i,can})_i \longrightarrow \frac{dt}{1+t} \quad i^{th} \text{ coordinate}$$

$(\omega_{i,can})$: basis of $Lie(\mathcal{G}/\mathbb{Z}[\![q^{S_n}]\!])^\vee$

$$f \in H^0(X, \underline{\omega}_V) \longrightarrow f(q) \in V_R \otimes \mathbb{Z}[\![q^{S_n}]\!].$$

$$f(q) = \sum_{T \in S_n} c_T(f) q^T$$

p-adic modular forms:

A/R abelian scheme, p is (topologically) nilpotent in R

Assume A is ordinary.

$$0 \longrightarrow A[p]^{mult} \longrightarrow A[p] \longrightarrow A[p]^{et} \longrightarrow 0$$

$A[p]^{et} \cong (\mathbb{Z}/p)^n$ locally

$A[p]^{et}$: étale of order p^n

$A[p]^{mult} \cong \mu_p^n$ locally for the étale topology.

(A,λ) $\quad \lambda: A \to {}^tA$.