

Let $f \in M_p^\infty(\Gamma)$ C^∞ -smooth Siegel modular form for

$\Gamma \subseteq \mathrm{Sp}_n(\mathbb{Z})$ and some (p, V) rep. of $\mathrm{GL}_n(\mathbb{C})$.

$$f: \mathfrak{H} \rightarrow V$$



$$\varphi = \varphi_f: \mathrm{Sp}_n(\mathbb{R}) \rightarrow V$$

$$\varphi(g) = \rho(j(g, i))^{-1} f(gi)$$

satisfying:

- $\varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in \Gamma$
- $\varphi(gk) = \underbrace{\rho(j(k, i))^{-1}}_{u(k)} \varphi(g) \quad \forall k \in K_\infty$
 \downarrow
 $j(k, i) = u(k)k(i)$

This gives a bijection between

$$M_p^\infty(\Gamma) \longleftrightarrow A_p(\Gamma \backslash \mathrm{Sp}_n(\mathbb{R}); V)$$

" space of such φ .

Question: How does one recognize the holomorphic Siegel modular forms in $A_p(\Gamma \backslash \mathrm{Sp}_n(\mathbb{R}); V)$?

Lie algebras:

$$\mathfrak{g} = \mathfrak{g}_n := \mathrm{Lie}(\mathrm{Sp}_n(\mathbb{R}))_{\mathbb{C}}$$

$$= \mathfrak{g}_n^- \oplus \mathfrak{k}_n \oplus \mathfrak{g}_n^+ \quad \text{Cartan decomp.}$$

$$\mathfrak{k}_n = \mathrm{Lie}(K_\infty)_{\mathbb{C}}$$

$$\mathfrak{g}_n^\pm := \left\{ \rho^\pm(\alpha) = \frac{1}{2} \begin{pmatrix} \alpha & \pm i\alpha \\ \pm i\alpha & -\alpha \end{pmatrix} : \alpha \in M_n(\mathbb{C}), \quad t\alpha = \alpha \right\}$$

Note that $[\mathfrak{g}_n^\pm, \mathfrak{g}_n^\mp] \subseteq \mathfrak{k}$

$$[\mathfrak{k}, \mathfrak{g}_n^\pm] \subseteq \mathfrak{g}_n^\pm$$

Adjoint action of K_∞

$$k = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_\infty$$

$$\text{Ad}(k) p^\pm(\alpha) = p^\pm((A \mp iB)\alpha^t (A \mp iB)).$$

$$\mathfrak{g}^+ \cong \text{Sym}^2(\mathbb{S}^n) \quad \text{as rep. of } K_\infty \cong U(n)$$

$$\mathfrak{g}^- \cong \text{Sym}^2(\mathbb{S}^n).$$

\mathfrak{g} acts on $C^\infty(Sp_{2n}(\mathbb{R}), \mathbb{V})$:

$$X \in \text{Lie}(Sp_{2n}(\mathbb{R}))$$

$$(X * \varphi)(g) = \left. \frac{d}{dt} \right|_{t=0} \varphi(g \exp(tX)).$$

This extends linearly to \mathfrak{g} , even to the universal enveloping algebra $U(\mathfrak{g})$.

For $f \in M_p^\infty(\mathbb{F})$, when is $f \in M_p(\mathbb{F})$, i.e., when is f holomorphic?

Then we see f is holomorphic $\Leftrightarrow X * \varphi_f = 0 \quad \forall X \in \mathfrak{g}^-$.

Example: $n=1$: $\mathfrak{g}^- = \mathbb{C} p^-(1)$, $p^-(1) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$
 $= \frac{1}{2} [\begin{pmatrix} 1 & -i \\ 0 & -i \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}]$

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \quad g(i) = x + iy = z.$$

$$(p^-(1) * \varphi_f)(g) = -iy^{k/2} \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{iff } f \text{ is holo.},$$

as claimed. ($p = x \mapsto x^k$)

More generally, $f \in M_k^\infty(\mathbb{F})$, we have

$f \in M_k^\infty(\mathbb{F})$ is holomorphic iff

$$f_{p(i) * \varphi_f} = -iy^2 \frac{df}{d\bar{z}} \in M_{k-2}^\infty(\mathbb{F})$$

$$f_{p(i) * \varphi_f} = iy^{-k} \frac{df}{d\bar{z}} (y^k f) \in M_{k+2}^\infty(\mathbb{F})$$

Maass-Shimura operator

Nearly holomorphic forms $\leftrightarrow \varphi \in A_p(\Gamma \backslash \mathbb{H}^n(\mathbb{R}); V)$ s.t.

there exists some r s.t.

$$\underbrace{\varphi^* * \varphi^* * \dots * \varphi^*}_{r \text{ copies}} * \varphi = 0$$

Take $z = (z_{ij}) \in \mathbb{H}^n$ $z_{ij} = x_{ij} + i y_{ij}$, $z_{ij} = z_{ji}$.

f nearly holomorphic $\Leftrightarrow f$ is a poly. in y_{ij} with holomorphic coefficients.

$$\begin{aligned} n=1: \quad \varepsilon * f &= -2\pi f_{\varphi^*(1) * \varphi} = 2\pi i y^2 \frac{df}{dz} \\ \delta * f &= -\frac{1}{2\pi} f_{\varphi^*(1) * \varphi} = \frac{-iy^k}{2\pi} \frac{d}{dz} (y^k f) \end{aligned}$$

$$\varepsilon \left(\frac{f}{(2\pi)^r} \right) = r \frac{f}{(2\pi y)^{r-1}} \quad f \text{ holomorphic}$$

Raising weights:

$$f \in M_r^\infty(\Gamma)$$

$$\downarrow$$

$$\varphi \in A_p^\infty(\Gamma; V) \longrightarrow D\varphi \in A_{p+1}^\infty(\Gamma, \text{Hom}(\varphi^*, V))$$

$\text{Hom}(\varphi^*, V)$

$$D\varphi(g)(X) := (X * \varphi)(g).$$

$$W \subseteq \mathcal{U}(\varphi^*) = \text{Sym}(\varphi^*)$$

$$\uparrow = \text{Sym}(\text{Sym}^2 \mathfrak{st}^V) \quad \text{as } K_\infty\text{-rep.}$$

f.d. K_∞ -sub-rep.

$$D_W \varphi \in A_{\rho \otimes W^V}(\Gamma; \text{Hom}(W, \rho))$$

$$D_W \varphi(g)(x) = (x * \varphi)(g) \quad \forall x \in W$$

$$x = \sum x_1 * x_2 * \dots * x_m, \quad x_i \in \mathfrak{g}^+$$

Don't get all possible reps. of K_{∞} .

Ex: $n=1$ Could only change the weight by even integers

$$\text{Sym}^k(x \mapsto x^{-2}) = \bigoplus_k (x \mapsto x^{-2k}).$$

The "double picture":

$$\mathfrak{h}_m \times \mathfrak{h}_m \hookrightarrow \mathfrak{h}_n \quad n=2m$$

$$(z, w) \longmapsto \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$$

$$\text{Sp}_{2m} \times \text{Sp}_{2m} \hookrightarrow \text{Sp}_{2n}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} a\alpha & b\beta \\ c\gamma & d\delta \end{pmatrix}$$

$$K_{\infty, 2m} \times K_{\infty, 2m} \hookrightarrow K_{\infty, 2n}$$

$$U(m) \times U(m) \hookrightarrow U(n)$$

$$(u_1, u_2) \longmapsto \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

$$\mathfrak{g}_n = \mathfrak{g}_n^- \oplus \mathfrak{k}_n \oplus \mathfrak{g}_n^+$$

$$\mathfrak{g}_n^+ = \mathfrak{p}_m^+ \oplus \mathfrak{p}_m^+ + \mathfrak{g} \quad (\text{similarly for } \mathfrak{g}_n^-)$$

$$\begin{array}{ccc} & & \swarrow \\ & & \mathfrak{p}^+ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \\ & & \alpha \in M_n(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathfrak{p}^+ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} & & \mathfrak{p}^+ \begin{pmatrix} 0 & 0 \\ \alpha & \alpha \end{pmatrix} \\ \alpha \in M_n(\mathbb{C}), \alpha \neq \alpha & & \end{array}$$

This is a $K_{\infty, n} \times K_{\infty, m}$ -decomp. (Ad-action)

$\mathfrak{g}' \cong \mathfrak{st}^v \oplus \mathfrak{st}^v$ as a $K_{\infty, m} \times K_{\infty, m}$ -rep.

$$\mathcal{U}(\mathfrak{g}') = \text{Sym}(\mathfrak{st}^v \oplus \mathfrak{st}^v) = \bigoplus_{\mu \geq 0} V_{\mu}^v \otimes V_{\mu}^v$$

$V_{\mu} =$ irred. rep. of highest wt. (μ_1, \dots, μ_n)

Start with $f \in M_{\det^k}(\Gamma)$, $\Gamma \subseteq \text{Sp}_{2n}(\mathbb{Z})$.

$\mu \geq 0$. Take $W = V_{\mu}^v \otimes V_{\mu}^v \subseteq \text{Sym}(\mathfrak{g}') = \mathcal{U}(\mathfrak{g}')$

$$(D_{\mu} \varphi_f)((g_1, g_2)) \in \text{Hom}(W, \det^k \otimes \det^k) \quad (g_1, g_2) \in \text{Sp}_{2m} \times \text{Sp}_{2n}$$

$$(D_{\mu} \varphi_f)((g_1, g_2))(X) = (X * \varphi_f)((g_1, g_2))$$

$$\begin{array}{c} \text{Hom}(W, \det^k \otimes \det^k) \\ \downarrow \text{project} \\ V_{\mu+k} \otimes V_{\mu+k} \end{array}$$

As we obtain

$$D_{\mu} f \in M_{k+\mu}^{\infty}(\Gamma') \otimes M_{k+\mu}^{\infty}(\Gamma'') \quad \Gamma' \times \Gamma'' \subseteq \Gamma$$

Even if f is holomorphic, $D_{\mu} f$ may only be nearly holo.

From this one can reach any weight.

Generally, one can write

$$D_{\mu} f = h_0 + \sum X_i * h_i$$

\uparrow
 holomorphic; $X_i \in \mathcal{U}(\mathfrak{p}_m^+ \times \mathfrak{p}_m^+)$
 h_i mod. forms.

The key will be to find a good f to start with.

In general: The good f 's are Siegel Eisenstein series.

Geometric picture:

$$\mathfrak{h}_n \ni \tau = (\tau_1, \dots, \tau_n)$$

$$I_n = (e_1, \dots, e_n)$$

$$A_{\tau} = \mathbb{C}^n / L_{\tau} \quad L_{\tau} = \mathbb{Z}\text{-span of } e_1, \dots, e_n, \tau_1, \dots, \tau_n$$

$$= \mathbb{Z}^n + \tau \mathbb{Z}^n$$

$$H_{\text{loc}}^1(A_{\tau}) = \text{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$$

$$\omega_{A_{\tau}} = \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$$

$$Y \in \text{Span}(\mathbb{Z})$$

$$A_{\tau} \longrightarrow A_{Y(\tau)}$$

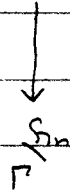
$$u \longmapsto {}^t j(Y, \tau)^{-1} u$$

$$A_{\tau}$$

$$Y(\tau) = (A\tau + B)(C\tau + D)^{-1}$$

$${}^t Y(\tau) = {}^t (C\tau + D)^{-1} \underbrace{{}^t (A\tau + B)}_{{}^t j(Y, \tau)}$$

$$\Gamma \backslash \mathbb{S}^n \times \text{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$$



$$\gamma(\tau, \lambda) = (\gamma(\tau), \lambda \cdot {}^t j(\gamma, \tau))$$

vector bundle \mathcal{H} over \mathbb{S}^n/Γ

Pullback to \mathbb{S}^n ,

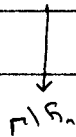
$$\alpha_i (\sum a_i e_i + \sum b_j \tau_j) = a_i$$

$$\beta_j (\sum a_i e_i + \sum b_j \tau_j) = b_j$$

↑
holo. trivialization over \mathbb{S}^n

$$C^\infty\text{-trivialization} \rightarrow du_1, \dots, du_n, d\bar{u}_1, \dots, d\bar{u}_n \quad du_i(u_j) = u_i$$

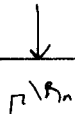
$$\Gamma \backslash \mathbb{S}^n \times \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})$$



$$\omega \subseteq \mathcal{H}$$

\mathbb{S}^n = standard rep. of GL_n

$$\omega \cong \Gamma \backslash \mathbb{S}^n \times V_{\mathbb{S}^n}$$



$$\gamma(\tau, v) = (\gamma(\tau), j(\gamma, \tau)v)$$

$n=1$: • holo. sections of $\omega^{\otimes k} \longleftrightarrow$ wt k modular forms

(ignoring the things needed to be compactified in the $n=1$ case)

- $\omega^{\otimes k} \subseteq \text{Sym}^k \mathcal{H}$
- Nearly holo. forms of wt $k \leftrightarrow$ holo. sections of $\text{Sym}^k \mathcal{H}$.

$$f(\tau) = \sum_{m=0}^k \underbrace{g_m(\tau)}_{\text{co-forms}} du^{k-m} d\bar{u}^m \quad f(\gamma(\tau)) = \gamma^* f(\tau)$$

$$\Rightarrow g_0(\gamma(\tau)) = j(\gamma, \tau)^k g_0(\tau)$$

$$f \rightarrow \varphi \in C^0(\rho^* \text{SL}_2(\mathbb{R}); \text{Sym}^k \mathcal{H})$$

$$\varphi(g) = g^{-1} f(g(u))$$

$$f(g(u)) = \sum_{m=0}^k \varphi_m(g) du^{k-m} d\bar{u}^m$$

$$f \text{ holo} \Leftrightarrow \varphi^{-1}(z) * \varphi = 0$$

$$\Leftrightarrow \varphi^{-1}(z) * \varphi_m = (m+1) \varphi_{m+1} \quad 0 \leq m < k$$

$$\varphi^{-1}(z) * \varphi_k = 0$$

$$\Rightarrow \underbrace{\varphi^{-1}(z) * \dots * \varphi^{-1}(z) * \varphi_0}_{k+1} = 0$$

$$\Rightarrow g_0 \text{ is nearly holomorphic.}$$