Let $f \in M_\rho^\infty(\Gamma)$ C^∞-smooth Siegel modular form for $\Gamma \leq Sp_{2n}(\mathbb{Z})$ and some (ρ, V) rep. of $GL_n(\mathbb{C})$.

$$f : \mathfrak{H}_n \to V$$

$$\rightsquigarrow$$

$$\varphi = \varphi_f : Sp_{2n}(\mathbb{R}) \longrightarrow V$$

$$\varphi(g) = \rho(j(g,i))^{-1} f(g(i))$$

satisfying:

- $\varphi(\gamma g) = \varphi(g) \quad \forall \gamma \in \Gamma$
- $\varphi(gk) = \rho(\underbrace{j(k,i)}_{u(k)})^{-1} \varphi(g) \qquad \forall\, k \in \underset{\substack{\text{I}\\ j(k,i) = u(k) \in U(n).}}{K_\infty}$

This gives a bijection between

$$M_\rho^\infty(\Gamma) \longleftrightarrow \underset{\text{space of such } \varphi.}{\mathcal{A}_\rho(\Gamma \backslash Sp_{2n}(\mathbb{R}); V)}$$

<u>Question:</u> How does one recognize the holomorphic Siegel modular forms in $\mathcal{A}_\rho(\Gamma \backslash Sp_{2n}(\mathbb{R}); V)$?

<u>Lie algebras:</u>

$$\mathfrak{g} = \mathfrak{g}_n := \mathrm{Lie}(Sp_{2n}(\mathbb{R}))_{\mathbb{C}}$$

$$= \mathfrak{p}_n^- \oplus \mathfrak{k}_n \oplus \mathfrak{p}_n^+ \quad \text{Cartan decomp.}$$

$\mathfrak{k}_n := \mathrm{Lie}(K_\infty)_{\mathbb{C}}$

$$\mathfrak{p}_n^\pm := \left\{ p^\pm(\alpha) = \frac{1}{2}\begin{pmatrix} \alpha & \pm i\alpha \\ \pm i\alpha & -\alpha \end{pmatrix} : \alpha \in M_n(\mathbb{C}),\ {}^t\alpha = \alpha \right\}.$$

Note that
$$[\mathfrak{p}^\pm, \mathfrak{p}^\mp] \subseteq \mathfrak{k}$$
$$[\mathfrak{k}, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm$$

Adjoint action of K_∞

$$k = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_\infty$$

$$Ad(k)\, p^{\pm}(\alpha) = p^{\pm}\left((A \mp iB)\, \alpha\, {}^t(A \mp iB)\right).$$

$\mathfrak{p}^+ \simeq Sym^2(st^\vee)$ as rep. of $K_\infty \cong U(n)$

$\mathfrak{p}^- \simeq Sym^2(st)$.

$\mathfrak{g}$ acts on $C^\infty(Sp_{2n}(\mathbb{R}), V)$:

$X \in Lie(Sp_{2n}(\mathbb{R}))$

$$(X * \varphi)(g) = \frac{d}{dt}\Big|_{t=0} \varphi(g \exp(tX)).$$

This extends linearly to $\mathfrak{g}$, even to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

For $f \in M_\rho^\infty(\Gamma)$, when is $f \in M_\rho(\Gamma)$, i.e., when is f holomorphic?

Then we see f is holomorphic $\Leftrightarrow X * \varphi_f = 0 \quad \forall X \in \mathfrak{p}^-$.

<u>Example</u>: $n=1$: $\mathfrak{p}^- = \mathbb{C}\, p^-(1)$, $p^-(1) = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$

$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$$

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \qquad g(i) = x + iy = z.$$

$$(p^-(1) * \varphi_f)(g) = -i y^{k/2} \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{iff} \quad f \text{ is holo.,}$$

as claimed. $(\rho = x \mapsto x^k)$

More generally, $f \in M_k^\infty(\Gamma)$, we have

$$f_{p^-(1) * \varphi_f} = -i y^2 \frac{df}{d\bar{z}} \in M_{k-2}^\infty(\Gamma)$$

$$f_{p^+(1) * \varphi_f} = i y^{-k} \frac{\partial f}{\partial z}(y^k f) \in M_{k+2}^\infty(\Gamma)$$

Maass - Shimura operator

Nearly holomorphic forms $\longleftrightarrow \varphi \in A_\rho(\Gamma \backslash Sp_{2n}(\mathbb{R}); V)$ s.t.

there exists some r s.t.

$$\underbrace{\mathfrak{p}^- * \mathfrak{p}^- * \cdots * \mathfrak{p}^-}_{r\text{-copies.}} * \varphi = 0$$

Take $z = (z_{ij}) \in \mathfrak{H}_n$ $\quad z_{ij} = x_{ij} + i\, y_{ij}$ $\quad , z_{ij} = z_{ji}.$

f nearly holomorphic $\Leftrightarrow$ f is a poly. in y_{ij}^{-1} with holomorphic coefficients.

$\underline{n=1:}$
$$\varepsilon * f = -2\pi\, f_{\mathfrak{p}^-(1) * \varphi_f} = 2\pi i y^2 \frac{df}{dz}$$
$$\delta * f = -\frac{1}{2\pi}\, f_{\mathfrak{p}^+(1) * \varphi_f} = \frac{-iy^{-k}}{2\pi} \frac{d}{dz}(y^k f)$$

$$\varepsilon\left(\frac{f}{(2\pi)^r}\right) = r\, \frac{f}{(2\pi y)^{r-1}} \qquad f \text{ holomorphic}$$

Raising weights:

$$f \in M_\rho^\infty(\Gamma)$$
$$\Big\downarrow$$
$$\varphi \in A_\rho^\infty(\Gamma; V) \longrightarrow D\varphi \in A^\infty_{\rho \otimes Sym^2(std)}(\Gamma, Hom(\mathfrak{p}^+, V))$$

$$D\varphi(g)(X) := (X * \varphi)(g).$$

$$W \subseteq \mathcal{U}(\mathfrak{p}^+) = Sym^\cdot(\mathfrak{p}^+)$$
$$= Sym^\cdot(Sym^2\, std^\vee) \quad \text{as } K_\infty\text{-rep.}$$

$\uparrow$

f.d. K_∞-sub-rep.

Skinner

6-16-10

pg 4

$D_W \varphi \in \mathcal{A}_{\rho \otimes W^\vee}(\Gamma; \mathrm{Hom}(W, \rho))$

$$D_W\varphi(g)(X) = (X * \varphi)(g) \qquad \forall X \in W$$

$$X = \sum X_1 * X_2 * \cdots * X_m, \quad X_i \in \mathfrak{p}^+$$

Don't get all possible reps. of K_∞.

Ex: $n=1$ Could only change the weight by even integers

$$\mathrm{Sym}^\cdot(x \mapsto x^{-2}) = \bigoplus_k (x \mapsto x^{-2k}).$$

The "double picture":

$$\mathfrak{H}_m \times \mathfrak{H}_m \hookrightarrow \mathfrak{H}_n \qquad n = 2m$$

$$(z, w) \longmapsto \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$$

$$Sp_{2m} \times Sp_{2m} \hookrightarrow Sp_{2n}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \left(\begin{array}{cc|cc} a & & b & \\ & \alpha & & \beta \\ \hline c & & d & \\ & \gamma & & \delta \end{array}\right)$$

$$K_{\infty,m} \times K_{\infty,m} \hookrightarrow K_{\infty,n}$$

$$U(m) \times U(m) \hookrightarrow U(n)$$

$$(u_1, u_2) \longmapsto \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$$

$$\mathfrak{g}_n = \mathfrak{p}_n^- \oplus \mathfrak{k}_n \oplus \mathfrak{p}_n^+$$

$$\mathfrak{p}_n^+ = \mathfrak{p}_m^+ \oplus \mathfrak{p}_m^+ + \mathfrak{p} \qquad \text{(similarly for } \mathfrak{p}_n^-\text{)}$$

$p^+\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ $\quad p^+\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$ $\quad p^+\begin{pmatrix} 0 & \alpha \\ {}^t\alpha & 0 \end{pmatrix}$

$\alpha \in M_n(\mathbb{C}),\ {}^t\alpha = \alpha$ $\qquad \alpha \in M_n(\mathbb{C})$

$$\underset{\cup}{K_{\infty,n}}$$

This is a $K_{\infty,m} \times K_{\infty,m}$ -decomp. (Ad-action)

$$\mathfrak{p}' \simeq St^\vee \otimes St^\vee \quad \text{as a } K_{\infty,m} \times K_{\infty,m} \text{-rep.}$$

$$\mathcal{U}(\mathfrak{p}') = \mathrm{Sym}^\cdot(St^\vee \otimes St^\vee) = \bigoplus_{\mu \geq 0} V_\mu^\vee \otimes V_\mu^\vee$$

V_μ = irred. rep. of highest wt. $(\mu_1, \dots, \mu_n)$

Start with $f \in M_{\det^\kappa}(\Gamma)$, $\Gamma \subseteq Sp_{2n}(\mathbb{Z})$.

$\mu \geq 0$. Take $W = V_\mu^\vee \otimes V_\mu^\vee \subseteq \mathrm{Sym}^\cdot(\mathfrak{p}') = \mathcal{U}(\mathfrak{p}')$

$$(D_\mu \varphi_f)((g_1, g_2)) \in \mathrm{Hom}(W, \det{}^\kappa \otimes \det{}^\kappa) \qquad (g_1,g_2) \in Sp_{2m} \times Sp_{2r}$$

$$(D_\mu \varphi_f)((g_1,g_2))(X) = (X * \varphi_f)((g_1,g_2))$$

$$\Big\downarrow \text{project}$$

$$V_{\mu+\kappa} \otimes V_{\mu+\kappa}$$

So we obtain

$$D_\mu f \in M^\infty_{\kappa+\mu}(\Gamma') \otimes M^\infty_{\kappa+\mu}(\Gamma'') \qquad \Gamma' \times \Gamma'' \subseteq \Gamma$$

Even if f is holomorphic, $D_\mu f$ may only be nearly holo.

From this one can reach any weight.

Generally, one can write

$$D_\mu f = h_0 + \sum X_i * h_i$$

h_0 holomorphic ; $X_i \in \mathcal{U}(\mathfrak{p}_m^+ \times \mathfrak{p}_m^+)$

h_i mod. forms.

The key will be to find a good f to start with.

In general : the good f's are Siegel Eisenstein series.

Geometric picture :

$\mathfrak{H}_n \ni \tau = (\tau_1, \ldots, \tau_n)$

$I_n = (e_1, \ldots, e_n)$

$A_\tau = \mathbb{C}^n / L_\tau \qquad L_\tau = \mathbb{Z}\text{-span of } e_1, \ldots, e_n, \tau_1, \ldots, \tau_n$

$\qquad = \mathbb{Z}^n + \tau\mathbb{Z}^n$

$$\begin{array}{ccc} H^1_{dR}(A_\tau) & = & \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C}) \\ \cup\!\! & & \cup\!\! \\ \omega_{A_\tau} & = & \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}) \end{array}$$

$\gamma \in Sp_{2n}(\mathbb{Z}) \qquad A_\tau \longrightarrow A_{\gamma(\tau)}$

$\qquad\qquad u \longmapsto {}^t j(\gamma, \tau)^{-1} u$

$$\gamma(\tau) = (A\tau + B)(C\tau + D)^{-1}$$
$$\overset{\shortparallel}{{}^t\gamma(\tau)} = \underbrace{{}^t(C\tau+D)^{-1}}_{{}^t j(\gamma,\tau)}\, {}^t(A\tau+B)$$

$$\begin{array}{c} {}_{\Gamma}\backslash^{\mathfrak{H}_n \times \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})} \\ \downarrow \\ {}_{\Gamma}\backslash^{\mathfrak{H}_n} \end{array} \qquad \gamma(\tau, \lambda) = (\gamma(\tau), \lambda \cdot {}^t j(\gamma, \tau)).$$

vector bundle $\mathcal{H}$ over $\mathfrak{H}_n/\Gamma$

Pullback to $\mathfrak{H}_n$, $\qquad \alpha_i(\sum a_i e_i + \sum b_j \tau_j) = a_i$

$\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \qquad \beta_j(\sum a_i e_i + \sum b_j \tau_j) = b_j$

holo. trivialization over $\mathfrak{H}_n$

C^∞-trivialization $\to du_1, \dots, du_n, d\bar{u}_1, \dots, d\bar{u}_n \qquad du_i((u_j)) = u_i$

$$\begin{array}{c} {}_{\Gamma}\backslash^{\mathfrak{H}_n \times \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C})} \\ \downarrow \\ {}_{\Gamma}\backslash \mathfrak{H}_n \end{array} \qquad \omega \leq \mathcal{H}$$

St = standard rep. of GL_n

$$\omega \simeq \begin{array}{c} {}_{\Gamma}\backslash^{\mathfrak{H}_n \times V_{St}} \\ \downarrow \\ {}_{\Gamma}\backslash \mathfrak{H}_n \end{array} \qquad \gamma(\tau, v) = (\gamma(\tau), j(\gamma, \tau) v)$$

<u>$n=1$:</u> • holo. sections of $\omega^{\otimes k} \longleftrightarrow$ wt k modular forms

(ignoring that things needed to be compactified in the $n=1$ case)

- $\omega^{\otimes k} \subseteq \mathrm{Sym}^k \mathcal{H}$
- Nearly holo. forms of wt k $\longleftrightarrow$ holo. sections of $\mathrm{Sym}^k \mathcal{H}$

$$f(\tau) = \sum_{m=0}^{k} \underbrace{g_m(\tau)}_{C^\infty\text{-forms}} du^{k-m} d\bar{u}^m \qquad f(\gamma(\tau)) = \gamma f(\tau)$$

$$\Rightarrow g_0(\gamma(\tau)) = j(\gamma,\tau)^k g_0(\tau)$$

$$f \longrightarrow \varphi \in C^\infty(\Gamma \backslash SL_2(\mathbb{R}); \mathrm{Sym}^k \mathcal{H})$$

$$\varphi(g) = g^{-1} f(g(i)) \qquad f(g(i)) = \sum_{m=0}^{k} \varphi_m(g)\, du^{k-m} d\bar{u}^m$$

$$f \text{ holo} \iff \wp^{-}(1) * \varphi = 0$$

$$\iff \wp^{-}(1) * \varphi_m = (m+1)\varphi_{m+1} \qquad 0 \le m < k$$

$$\wp^{-}(1) * \varphi_k = 0$$

$$\Rightarrow \underbrace{\wp^{-}(1) * \cdots * \wp^{-}(1)}_{k+1} * \varphi_0 = 0$$

$\Rightarrow g_0$ is nearly holomorphic.