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Pull-back formulas, differential operators and construction of p-adic families of holomorphic cusp forms for unitary groups:

Goal of this series of talks: Explain a construction of p-adic families of holomorphic and nearly holomorphic Siegel modular forms. (The techniques work in great generality, just restricting to Siegel modular forms to simplify the notation.)

Example of a p-adic family: $\zeta_p(s) = \dfrac{g((1+p)^s-1)}{\underbrace{(1+p)^s-(1+p)}}$

where $g(T) \in \mathbb{Z}_p[\![T]\!] = \Lambda$. $\quad h((1+p)^s-1)$

$h(T) = 1+T-(1+p)$.

$$E = \frac{1}{2}\,\frac{g((1+p)(1+T)^{-1}-1)}{h((1+p)(1+T)^{-1}-1)} + \sum_{n=1}^{\infty}\Big(\sum_{\substack{d\mid n\\ p\nmid d}} d^{-1}(1+T)^{r_d}\Big)q^n \in \Lambda[\![q]\!]$$

p odd for simplicity. $\qquad d = \omega(d)(1+p)^{r_d}$

$\mathbb{Z}_p^\times = \mu_{p-1} \times (1+p)^{\mathbb{Z}_p}$

Put $1+T = (1+p)^k \quad k \geq 2$.

Resulting q-expansion is that of $E_k^{\mathrm{ord}}(\omega^{-k}) =$ (i.e, has U_p-eigenvalue $= 1$.)

E is a p-adic family in the sense that its q-expansion coeffs are p-adic analytic functions and its specializations at "classical weights" are q-expansions of classical modular forms.

Weight space: $\mathcal{W} = \mathrm{Hom}_{cts}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times) \hookleftarrow \mathbb{Z} \; : x \mapsto x^k$

$$\shortparallel$$

$$\mathrm{Hom}_{cts}(\mu_{p-1} \times (1+p)^{\mathbb{Z}_p}, \mathbb{C}_p^\times) \qquad \varphi$$

$$\mathbb{Z}/(p-1)\mathbb{Z} \times \{\alpha \in \mathbb{C}_p^\times : |\alpha - 1|_p < 1\} \qquad \varphi(\zeta(1+p)^a) = \zeta^m \alpha^a.$$

$$\overset{\in}{m}$$

$k_0 \in \mathbb{Z}_{>0}$. $\mathcal{U} \ni k_0$ a nbd in $\mathcal{W}$ (affinoid nbd perhaps)

ex: $\mathcal{U}_r = k_0 \times \{\alpha \in \mathbb{C}_p^\times : |\alpha - (1+p)^k|_p < p^{-r}\}$.

$\mathcal{O}(\mathcal{U}_r) =$ analytic functions on $\mathcal{U}_r / \mathbb{Q}_p \simeq \mathbb{Q}_p\langle p^{-r}T\rangle$

Then consider formal expansions

$$F = \sum_{n=0}^{\infty} a_n q^n \quad , \quad a_n \in A \qquad A/\mathcal{O}(\mathcal{U}) \text{ finite extension}$$

$\phi \in \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}, cts}(A, \overline{\mathbb{Q}}_p)$ s.t. $\phi|_{\mathcal{O}(\mathcal{U})} =$ specialization at $\alpha = (1+p)^k$

require the resulting q-expansion

$$F_\phi = \sum_{n=0}^{\infty} \phi(a_n) q^n$$

to be the q-expansion of a wt-k modular form.

Question: Does any form f of wt k_0 fit into such a family?

(eigenform $f \rightsquigarrow$ eigenfamily)

$$f\, \mathbb{E}^*((1+p)^{-k_0}(1+T)) = G$$

$$\mathbb{E}^* = h((1+p)(1+T)^{-1})\, \mathbb{E}$$

wt $k=0$ specialization of $\mathbb{E}^* = $ const.

Then the weight k_0 specialization of G is (const) $\cdot f$. This shows the answer to the first question is yes, but this does not answer the second question. The second question is harder! This one is most easily dealt with by focusing on families of fixed finite slopes ($|U_p\text{-eigenvalue}|_p = \text{constant} \neq 0$)

Ex: E U_p- eigenvalue $=1$.

For finite slope families:

$$M^r(A) = A\text{-module of } A\text{-families of slope } r = \mathrm{ord}_p(U_p\text{-eigenvalue}) < \infty$$

This is a finite (torsion-free) A-module. One can act on this by Hecke-alg. $/A$. Can find eigenprojectors.

Note: When $r=0$, this is the ordinary case and has been developed by Hida. $r>0 \rightsquigarrow$ Coleman, ...

By work of Ash-Stevens, Urban can construct a projector to slope-r forms over some nbd $\mathcal{U} \ni k_0$. If we assume f has slope-r, we can put it into a family of slope-r forms and then into an eigenfamily.

Can imagine generalizing this:

(i) def. of families of forms ✓

(ii) construct finite slope projectors ✓

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(iii) put a given finite slope form into an analytic family

ideally: family should have as much freedom as weights allow

ex: F totally real deg d H.M.F $-(k_1, .., k_d) \in \mathbb{Z}^d$

problem E.S $-(k, \dots, k)$

$\Rightarrow$ can only move E.S in one direction.

Same problem for Siegel modular forms.

$$\mathbb{E}^* = \sum_{n=0}^{\infty} a(n) q^n \qquad a(n) \in \Lambda$$

some analytic fctn.

$$\sum_{n=1}^{\infty} n a(n) q^n + \frac{\log(1+T)}{2\pi y} \mathbb{E}^*((1+p)^{-2}(1+T))$$

Specialize at wt k : get the q-exp. $(k>2)$

$$\underbrace{\frac{1}{2\pi} y^{2-k} \frac{d}{dz}}_{\text{Maass-Shimura differential operator: takes wt } k \text{ to wt } k+2}(y^{k-2} \mathbb{E}_{k-2}^*)$$

$k=2$: (multiple) $E_2^{\text{cont}}(z) = E_2(z) - E_2(pz)$.

This is an example of a family of nearly-holomorphic modular forms of slope = 1.

We deal with part (iii) by restricting nearly-holomorphic families from a larger group.

Ex: $\mathfrak{h} \times \mathfrak{h} \hookrightarrow \mathfrak{h}_2$

Siegel Modular Forms:

The symplectic (similitude) group:

$n \geq 1$

$J = J_n = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}$

$GSp_{2n} = \{ g \in GL_{2n} : g J {}^t g = \lambda_g J \} \qquad \lambda_g \in \mathbb{G}_m$

$\lambda : GSp_{2n} \longrightarrow \mathbb{G}_m$ similitude character.

$g \longmapsto \lambda_g$

$Sp_{2n} = \ker \lambda$.

Ex: $n=1$ $\quad GSp_2 = GL_2$

$Sp_2 = SL_2$.

$\lambda = \det$.

$Sp_{2n}(\mathbb{R})$

$\cup|$

$K_\infty = K_{\infty,n} = \{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : Sp_{2n}(\mathbb{R}) \}$ maximal compact

$K_\infty \xrightarrow{\sim} U(n)$

$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \longmapsto u(k) = A + iB$.

$\shortparallel$

$k(A,B) = k$

Ex: $n=1$

$K_\infty = SO_2(\mathbb{R}) \simeq U(1)$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\theta}$.

The half-space $\quad \mathfrak{H} = \mathfrak{H}^n = \{ Z \in M_n(\mathbb{C}) : {}^t Z = Z,\ Z = X + iY,\ Y > 0 \}$

$Sp_{2n}(\mathbb{R})$ acts on $\mathfrak{H}^n$ via

$$\underset{\gamma}{\begin{pmatrix} A & B \\ C & D \end{pmatrix}} \cdot Z = \gamma(Z) = (AZ+B)(CZ+D)^{-1}.$$

- transitive action
- K_∞ = stabilizer of $i \in \mathfrak{H}$
- $Sp_{2n}(\mathbb{R})/K_\infty \longrightarrow \mathfrak{H}$ $\quad \gamma \mapsto \gamma(i)$.

The automorphy factor:

$$j : Sp_{2n}(\mathbb{R}) \times \mathfrak{H} \longrightarrow U(n)_{\mathbb{C}} = GL_n(\mathbb{C})$$

$$j(\gamma, Z) = CZ + D \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Weights: complex algebraic representations of $GL_n(\mathbb{C})$: $\quad (\rho, V)$

irred Indexed by highest weights wrt upper triangular Borel

$$\rho \longleftrightarrow (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$$
$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$$

action of $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$ on highest weight vector $\prod t_i^{\mu_i}$:

Siegel modular forms: $\quad (\rho, V)$ weight

$\Gamma \subseteq Sp_{2n}(\mathbb{Z})$ a congruence subgroup (i.e. Γ contains $\{\gamma \equiv 1 \bmod N\}$ for some $N \geq 1$).

C^∞-case: smooth functions $f : \mathfrak{H} \longrightarrow V$ s.t.

$$(f|_\rho \gamma) := \rho(j(\gamma, z))^{-1} f(\gamma(z))$$
$$= f \quad \forall\ \gamma \in \Gamma.$$

Ex: $n=1$ $\quad \rho = \det^{k} = x^{k}$

recover usual def.

holomorphic case just look at holomorphic functions instead of smooth.

(we ignore any growth conditions here)

Fourier expansions: $\quad \Gamma \supseteq \Gamma(N) \Rightarrow \Gamma \supseteq \begin{pmatrix} 1 & NX \\ 0 & 1 \end{pmatrix} \quad {}^tX = X \in M_n(\mathbb{Z})$.

$$\Rightarrow F = \sum_{\substack{T \geq 0 \\ {}^tT = T \in L \subseteq \frac{1}{2N} M_n(\mathbb{Z})}} c(T)\, e(\mathrm{tr}(TZ)) \qquad c(T) \in V.$$

lattice

Hecke operators: Can find classical info. in Andrianov's books.

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} \qquad d_i \text{ integers} \quad d_1 | d_2 | \cdots | d_n$$

$$T_D = \Gamma \begin{pmatrix} D^{-1} & \\ & D \end{pmatrix} \Gamma = \coprod \Gamma \alpha_i$$

$$T_D f = \left(\prod d_i^{\mu_i - (n+1)} \right) \sum_i f|_{\rho} \alpha_i$$

$$\rho \longleftrightarrow (\mu_1, \ldots, \mu_n)$$

Ex: $n=1$.

$$T_\ell = \Gamma \begin{pmatrix} \ell^{-1} & \\ & \ell \end{pmatrix} \Gamma \qquad \ell \nmid N.$$

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$$T_\ell f = \ell^{2k-2} f(\ell^2 z) + \frac{1}{\ell^2} \sum_{a=1}^{\ell^2} f\left(\frac{z+a}{\ell^2}\right).$$