Sharifi

Galois Cohomology, Iwasawa theory, and p-adic L-functions:

G topological group

M topological G-module

$C^i(G,M) = \{ f : G^i \to M \mid f \text{ cts} \}$

$H^i(G,M) = H^i(C^\bullet(G,M))$.

$A = \varinjlim A_\alpha$ $\quad$ G-submodules

$H^i(G,A) \cong \varinjlim H^i(G, A_\alpha)$.

$T \cong \varprojlim T_n$ $\quad$ T_n finite, discrete $\mathbb{Z}[G]$-module

$$0 \longrightarrow \varprojlim{}^1 H^{i-1}(G,T_n) \longrightarrow H^i(G,T) \longrightarrow \varprojlim H^i(G,T_n) \longrightarrow 0$$

this will be 0 if $H^{i}(G,M)$ is finite $\forall$ finite, discrete G-module M. $(*)$

Local fields: F local field, non-arch. char $\neq p$. F_s = seperable closure of F.

$G_F = Gal(F_s/F)$ absolute Galois grp.

G_F satisfies $(*)$ for finite $\mathbb{Z}_p[G]$-modules.

The p-cohom. dim. of G_F is 2.

Kummer Theory:

$$1 \longrightarrow \mu_{p^n} \longrightarrow F_s^\times \xrightarrow{p^n} F_s^\times \longrightarrow 0$$

$$H^1(G_F, \mu_{p^n}) \cong F^\times / (F^\times)^{p^n}$$

$$H^2(G_F, \mu_{p^n}) = Br(F)[p^n] \underset{\cong}{\xrightarrow{\text{Inv}}} \tfrac{1}{p^n}\mathbb{Z}/\mathbb{Z}$$

$$H^1(G_F, \mathbb{Z}_p(1)) \cong \varprojlim H^1(G_F, \mu_{p^n})$$
$$\cong (F^\times)^{(p)} = p\text{-completion of } F^\times$$

$$H^2(G_F, \mathbb{Z}_p(1)) \cong T_p(Br(F)) \ (= \text{Tate module})$$
$$\cong \mathbb{Z}_p.$$

$$\underset{\substack{\cup \\ \pi \text{ quot. map}}}{H^1(G_F, G_F^{ab,(p)})} \otimes_{\mathbb{Z}_p} H^1(G_F, \mathbb{Z}_p(1)) \xrightarrow{\cup} H^2(G_F, G_F^{ab,(p)}(1)).$$
$$\cong H^2(G_F, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} G_F^{ab,(p)}$$
$\uparrow$ b/c cohom dim $= 2$.
$$\cong G_F^{ab,(p)}$$

$$\rho_F^{(p)} : (F^\times)^{(p)} \xrightarrow{\sim} G_F^{ab,(p)}$$
$$a \longmapsto \pi \cup a$$
This gives a construction of the local reciprocity map.

Tate duality: T compact $\mathbb{Z}_p[G_F]$-module s.t. $T \cong \varprojlim_n T_n$ where T_n is a finite $\mathbb{Z}_p[G_F]$-module.

$T^\vee = \mathrm{Hom}_{cts}(T, \mathbb{Q}_p/\mathbb{Z}_p)$ Pontryagin dual. It's discrete.

$$H^i(G_F, T) \times H^{2-i}(G_F, T^\vee(1)) \xrightarrow{\cup} H^2(G_F, \mu_{p^\infty}).$$
$$\cong \mathbb{Q}_p/\mathbb{Z}_p$$
is a cts. nondeg. pairing.

$$\rightsquigarrow H^i(G_F, T) \cong H^{2-i}(G_F, T^\vee(1))^\vee.$$

To get the reciprocity map here: Take $T = \mathbb{Z}_p(1)$
$\Rightarrow T^\vee(1) \cong \mathbb{Q}_p/\mathbb{Z}_p$

$i = 1$:
$$\begin{array}{ccc} H^1(G_F, \mathbb{Z}_p(1)) & \longrightarrow & H^1(G_F, \mathbb{Q}_p/\mathbb{Z}_p)^\vee \\ \wr\Vert & & \Vert \\ (F^\times)^{(p)} & \xrightarrow[\sim]{\rho_F^{(p)}} & G_F^{ab,(p)}. \end{array}$$

<u>Global fields:</u> F global field, $\mathrm{char}(F) \neq p$. S is a finite set of primes of F including (if F is a # field) all primes over p and all real places.

Assume that $p \neq 2$ if F has real places.

$G_{F,S}$ = Galois group of the maximal unram. outside S ext. Ω of F.

The assumption gives $\mathrm{cd}_p\, G_{F,S} \leq 2$ & $G_{F,S}$ satisfies (#).

<u>Kummer theory:</u>

$$1 \to \mu_{p^n} \longrightarrow \mathcal{O}_{\Omega,S}^{\times} \xrightarrow{p^n} \mathcal{O}_{\Omega,S}^{\times} \longrightarrow 1$$

units in $\mathcal{O}_{\Omega,S}$ = ring of S-ints of Ω

$= \{a \in \Omega : v_{\wp}(a) \geq 0 \ \forall \wp \notin S_\Omega =: S\}$

$$H^0(G_{F,S}, \mathcal{O}_{\Omega,S}^{\times}) = \mathcal{O}_{F,S}^{\times}$$

$$H^1(G_{F,S}, \mathcal{O}_{\Omega,S}^{\times}) \cong Cl_{F,S} = Cl_F / \langle [\wp] \mid \wp \in S \rangle$$

For $[\mathfrak{a}] \in Cl_{F,S}$ $\quad \mathfrak{a}\mathcal{O}_{\Omega,S} = \alpha \mathcal{O}_{\Omega,S}$

$$\downarrow$$

$(\sigma \mapsto \alpha^{\sigma - 1}) \in H^1(G_{F,S}, \mathcal{O}_{\Omega,S}^{\times})$. This gives the isom.

$$H^2(G_{F,S}, \mathcal{O}_{\Omega,S}^{\times})[p^\infty] \cong Br_S(F)[p^\infty]$$

$$Br_S(F) = \mathrm{Ker}\Big(\bigoplus_{v \in S} Br(F_v) \xrightarrow{\Sigma\, \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \Big).$$