X smooth projective variety $/\mathbb{C}$, dim n

$$\begin{array}{ccl} CH^k(X)_0 & \xrightarrow{AJ} & J^k(X) = Fil^{n-k+1}\, H^{2n-2k+1}(X,\mathbb{C})^\vee / H_{2n-2k+1}(X,\mathbb{R}) \\ | & & \quad \cup \\ CH^k(X)_{alg} & \xrightarrow{AJ} & J_a^k(X) = \text{largest subtorus whose tangent space} \\ & & \qquad\qquad \text{is contained in } H^{n-k+1,n-k}(X)^\vee \end{array}$$

$CH^k(X)_0 / CH^k(X)_{alg} =: Gr^k(X)$

(i) $Gr^k(X)$ is at most countable

(ii) But $Gr^k(X) \otimes \mathbb{Q}$ can be infinite dimensional.

(iii) AJ is in general neither injective or surjective.

Schoen: $\widetilde{W}_N$: Kuga-Sato threefold over $X(N)$

$$\begin{tikzcd} \mathcal{E} \arrow[d] & \mathcal{E} \times_{Y(N)} \mathcal{E} \arrow[r, hook] \arrow[d] & \widetilde{W}_N \arrow[d] \\ Y(N) & Y(N) \arrow[r] & X(N) \end{tikzcd}$$

$Gr^2(\widetilde{W}_{N_{\bar{\mathbb{Q}}}}) \otimes \mathbb{Q}$ is infinite dimensional.

Nori: For $k \geq 3$, it can happen that $AJ = 0$, but Gr^k is not torsion.

Bloch-Beilinson Conjecture: Let K be a number field, X/K smooth projective.

(i) $CH^k(X)_0$ is a finitely generated abelian group.

(ii) Both AJ and $AJ_{et,\ell}$ are both injective mod torsion.

Prasanna
6-17-10
pg 2

$$\begin{tikzcd} & J^k(X) \\ CH^k(X)_0 \arrow[ur, "AJ"] \arrow[dr, "AJ_{et,p}"'] & \\ & H^1(G_K, H^{2k-1}(\bar{X}, \mathbb{Q}_p)(k)) \end{tikzcd}$$

$\bar{X} = X \otimes_K \bar{K}$.

(iii) $\operatorname{rank} CH^k(X)_0 = \operatorname{ord}_{s=k} L(H^{2k-1}(\bar{X}), s)$

$= \operatorname{ord}_{s=0} L(H^{2k-1}(\bar{X})(k), s)$.

Example: Let $C: x^4+y^4+z^4=0$.

$X = \mathrm{Jac}(C) \underset{\text{isog.}}{\sim} E\times E\times E$, $E: y^2 = x^3 - x \longleftrightarrow$ A Hecke char. ψ of $K=\mathbb{Q}(i)$ since E has CM by $\mathbb{Z}[i]$.

One can compute that

$\operatorname{rk}(CH^2(X)_0) = \operatorname{ord}_{s=2} L(H^3(\bar{X}), s)$

$L(H^3(\bar{X}), s) = L(\psi^3, s)\, L(\psi, s-1)^9$

$\qquad\qquad - \qquad\qquad + \qquad \leftarrow$ sign of fctn. eq.

Might expect that one should get a nontorsion cycle in $Gr^2(X)$.

In this case, Bloch showed that $\Delta_C := C - [-1]^*C$ is nontorsion in $Gr^2(X)$.

"Heights of Δ_C" $\longleftrightarrow L'(\psi^3, 2)$.

Example: Let K be an imaginary quadratic field with

(i) $h_K = 1$

(ii) roots of unity in $K = \{\pm 1\}$

(iii) odd disc. $= -D$

$$\mathcal{O}_K/\sqrt{-D} \xrightarrow{\sim} \mathbb{Z}/D\mathbb{Z}$$

$$(\mathcal{O}_K/\sqrt{-D})^\times \longrightarrow (\mathbb{Z}/D\mathbb{Z})^\times \xrightarrow{\varepsilon_K} \{\pm 1\}$$

Define a Hecke character ψ of K

$\psi((\alpha)) = \varepsilon_K(\alpha)\cdot\alpha$, type $(1,0)$

$\psi^2((\alpha)) = \alpha^2$ type $(2,0)$

ψ^2 is an unramified character.

$$f = \Theta_{\psi^2}(z) = \sum_{\sigma\in\mathcal{O}_K} \psi^2(\sigma)\, e^{2\pi i (N\sigma) z} \in S_3(\Gamma_0(D), \varepsilon_K)$$

$\Downarrow$

Hecke eigenform

Associated Galois representation lives in $H^2(W_1)$ where W_1 is Kuga-Sato surface over $X_1(D)$.

Also, we can associate to f (Shioda) an element in $H^{3,0}(W_1)$:

$$\begin{array}{ccc} (z,t)\in \mathfrak{H}\times\mathbb{C}/_{\Gamma_1(D)\ltimes\mathbb{Z}^2} = \mathcal{A} & \longrightarrow & W_1 \\ \downarrow & & \downarrow \\ Y_1(D) & \longrightarrow & X_1(D) \end{array}$$

$$\omega_f = f(z)\, dz\wedge dt \in H^{2,0}(W_1)$$

$\psi \longleftrightarrow$ elliptic curve $A/\mathbb{Q}$, CM by $\mathcal{O}_K$.

$X_1 = A \times W_1$ is a threefold.

$CH^2(X_1)_0 \qquad H^3(X_1) \supseteq H^1(A) \otimes H^2(W_1)$

$L(\mathrm{Ind}_K^{\mathbb{Q}}(\psi) \otimes \mathrm{Ind}_K^{\mathbb{Q}}(\psi^2), s) = L(\psi^3, s)\, L(\psi, s-1) \qquad \text{sign} = -1.$

Bloch - Beilinson predicts there should be a cycle:

$\mathrm{diag} - A\times 0 - 0 \times A =: \Delta_1.$

$\cap$

$W_1 \to X_1(D)$, fiber $= A$ over $\cdot$ Heegner pt assoc. to K

$A \times W_1 \supseteq A \times A \to X_1(D)$

Variant: r odd pos. integer, ψ^{r+1} is unramified

$f = \Theta_{\psi^{r+1}} \in S_{r+2}(\Gamma_0(D), \varepsilon_K)$

$\downarrow$

$\omega_f \in H^{r+1,0}(W_r)$

W_r = Kuga - Sato variety of dim $r+1$ over $X_1(D)$.

$X_r = A^r \times W_r$

$H^{2r+1}(X_r) \supseteq H^r(A^3) \otimes H^{r+1}(W_r)$

L-function will contain a factor: $L(\psi^{2r+1}, s)\, L(\psi, s-r)$

sign $= (-1)$

Case 1: sign $L(\psi^{2r+1}, s) = 1$, sign $L(\psi, s-r) = -1$

Case 2: " $= -1$, " $= 1$

Note $A^r \times A^r \subseteq A^r \times W_r$.

$\Delta_r := (\text{diag} - A\times 0 - 0\times A)^r \subseteq A^r \times A^r$.

<u>Claim:</u> Δ_r is homologically trivial. (actually needs to be modified a little, but ignore that for this talk).

Case 2 is where applications to the Griffiths group are.

<u>Joint work w/ Bertolini & Darmon:</u>

<u>Thm:</u> Suppose we are in case 2 and $L(\psi, 1) \neq 0$. Let p be a prime that is split in K. Then $\forall$ integers $r >> 0$, $r \equiv 0 \pmod{p-1}$,

$$[\Delta_r] \in Gr^{r+1}(X_r) \quad \text{is non-torsion.}$$

<u>Idea:</u> Compute the image of Δ_r under the p-adic Abel-Jacobi map.

X = variety / number field L, $\dim X = n$; $k+j=n$.
$Z \in Z^k(X)_0$, $\dim Z = j$. $U = X - Z$.
Gysin-sequence:

$$\begin{tikzcd} H_{2j+1}(Z) \arrow[r] \arrow[d, phantom, "\parallel"] & H_{2j+1}(X) \arrow[r] & H_{2j+1}(U) \arrow[r] & H_{2j}(Z) \arrow[r] & H_{2j}(X) \\ 0 & & & & {[Z]} \\ 0 \arrow[r] & H_{2j+1}(X) \arrow[r] & V \arrow[r] & \mathbb{Q}_p \arrow[uu] & 1 \arrow[u, mapsto] \end{tikzcd}$$

Prasanna

6-17-10

p96

$$CH^k(X_L)_0 \xrightarrow{AJ_{et,p}} H^1(L, H^{2k-1}_{et}(\bar{X}, \mathbb{Q}_p)(k))$$

Let p be a prime split in K

$$\begin{tikzcd}
CH^{r+1}(X_r)_{0/K} \arrow[r] \arrow[ddr, "AJ_p"] & H^1(K, H^{2r+1}(\bar{X}_r)(r+1)) \arrow[d] \\
 & H^1(\mathbb{Q}_p, H^{2r+1}(\bar{X}_r)(r+1)) \\
\mathrm{Fil}^{r+1} H^{2r+1}_{dR}(X_r)^\vee \arrow[r, "\sim"', "\exp"] & H^1_f(\mathbb{Q}_p, H^{2r+1}(\bar{X}_r)(r+1)) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

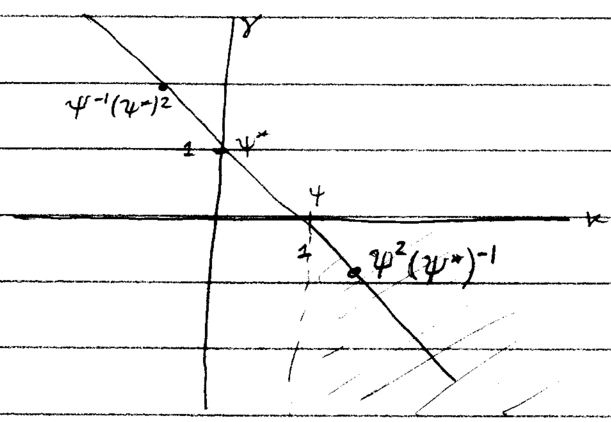
$\uparrow$

Bloch-Kato exp.

$X_r = A^r \times W_r$

$\omega_A^r \wedge \omega_f \quad (2r+1, 0)$

<u>Katz p-adic L-function</u>



$\psi^* = \psi \circ$ (complex conj of K)

There is a p-adic L-function $\mathcal{L}_p$ satisfying

$$\frac{\mathcal{L}_p(\psi^k(\psi^*)^j \chi)}{\Omega_p^{(\,)}} \overset{(*)}{=} \frac{L(\psi^{-k}(\psi^*)^{-j}\chi^{-1}, 0)}{\Omega_\infty^{(\,)}}$$

fin. order

Prasanna
6-17-10
PS 7

for every (k,j) in the shaded region.

Thm: $[AJ_p(\Delta_r)(w_A^r \wedge w_f)]^2 = (*) \underbrace{\mathcal{L}_p(\psi)}_{= L(\psi,1)} \cdot \mathcal{L}_p(\psi^{-r}(\psi^*)^{r+1})$

$\Rightarrow$ $r >> 0$, $r \equiv 0 \pmod{p-1}$, $\mathcal{L}_p(\psi^{-r}(\psi^*)^{r+1}) \neq 0$

$\Rightarrow$ $[\Delta_r]$ is nontorsion in $Gr^{r+1}(X_r)$.

Case 1: $X_1 = A \times W_1$

$$\begin{array}{ccc} CH^2(X_1)_0 & \longrightarrow & J^3(X_1) \\ \Big| & & \cup \\ CH^2(X_1)_{alg} & \longrightarrow & J_a^3(X_1) = A \end{array}$$

$$\begin{array}{c} A \times X_1 \\ \parallel \\ A \times (A \times W_1) \\ \parallel \\ (A \times A) \times W_1 \\ \psi^2 \end{array}$$

ψ^2 occurs in $A \times A$ and W_1.

Tate conj. / Hodge conj $\Rightarrow$ $\exists$ a correspondence that realizes this.
Unfortunately, this is unknown. So not much is known here...