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p-adic L-functions and the Griffiths group:

X smooth projective variety $/\mathbb{C}$, $n = \dim X$

Question: What can be said about all possible subvarieties of X?

Ex: $n=1$ X = a curve

$Z^1(X)$ = free abelian group on points of X.
$= \{\sum n_P P : n_P \in \mathbb{Z}\}$.

$$Z^1(X) \xrightarrow{\deg} \mathbb{Z} \qquad Z^1(X)_0 = \ker(\deg)$$
$$\sum n_P P \longmapsto \sum n_P.$$

$$0 \to Z^1(X)_0 \to Z^1(X) \to \mathbb{Z} \to 0.$$

$$0 \to \underset{\{\mathrm{div}(f)\,:\, f \text{ a rat. fctn on } X\}}{\underset{\parallel}{Z^1(X)_{rat}}} \to Z^1(X)_0 \to Z^1(X)_0 / Z^1(X)_{rat} \to 0$$

$\mathrm{Jac}(X) = (\Omega^1(X))^\vee / H_1(X,\mathbb{Z})$ = a complex torus of dim g.
where $g = \mathrm{genus}(X)$. This is a principally polarized abelian variety.

$$Z_1(X)_0 \xrightarrow{AJ} \mathrm{Jac}(X)$$
$$\sum_P n_P P \longmapsto \left(\omega \longmapsto \sum_P n_P \int_{P_0}^{P} \omega\right) \qquad P_0 = \text{base pt}.$$

Thm (Abel-Jacobi): (i) $\ker(AJ) = Z^1(X)_{rat}$

(ii) AJ is surjective.

So one has

$$Z^1(X)_0 / Z^1(X)_{rat} \xrightarrow{\sim} \mathrm{Jac}(X).$$

We return to the general case.

<u>Def:</u> $Z^k(X)$ = free abelian group on closed irred. subvarieties of X of codim k. Elements in $Z^k(X)$ will be called "cycles."

<u>Analogue of the degree map:</u>

Cycle class map: $cl : Z^k(X) \longrightarrow H^{2k}(X, \mathbb{Z})$

$Z \in Z^k(X)$, suppose Z is smooth irred.

$$\mathbb{Z} \xrightarrow{\sim} H_{2n-2k}(Z) \longrightarrow H_{2n-2k}(X) \xrightarrow{\text{Poin. Duality}} H^{2k}(X, \mathbb{Z})$$

$$1 \longmapsto cl(Z).$$

<u>Def:</u> $Z^k(X)_0$ = ker of $cl : Z^k(X) \to H^{2k}(X, \mathbb{Z})$.

= "cycles homologically equivalent to zero"

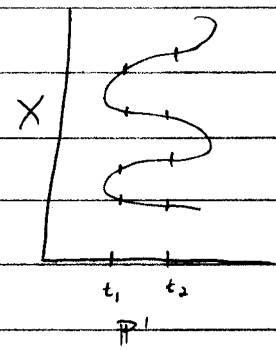
<u>Def:</u> A cycle $Z \in Z^k(X)$ is said to be rationally equivalent to 0 if $\exists$ a collection of subvarieties Y_i in X of codim $k-1$ and rational functions f_i on Y_i such that

$$Z = \sum_i \mathrm{div}(f_i).$$

$Z_1 \sim_{rat} Z_2$ if $Z_1 - Z_2 \sim_{rat} 0$.

This def. is equivalent to the following:

Z_1, Z_2 are rationally equivalent if $\exists\ Y \in Z^k(X \times \mathbb{P}^1)$, and two points $t_1, t_2 \in \mathbb{P}^1$ s.t. $Z_1 = Y_{(t_1)}$ $Z_2 = Y_{(t_2)}$



(Fulton's book on intersection theory)

Def: (alg. equivalence) Same definition as rational equivalence except replace $\mathbb{P}^1$ by "a smooth connected variety."

$$Z^k(X)_{rat} \subseteq Z^k(X)_{alg} \subseteq Z^k(X)_0 \subseteq Z^k(X).$$

$$CH^k(X) = Z^k(X)/Z^k(X)_{rat}$$

$$CH^k(X)_0 = Z^k(X)_0/Z^k(X)_{rat}$$

$$CH^k(X)_{alg} = Z^k(X)/Z^k(X)_{alg}.$$

Ex: (Divisors) n is arbitrary, $k=1$.

$$D \longmapsto \mathcal{L}(D)$$

$$D_1 \sim_{rat} D_2 \iff \mathcal{L}(D_1) \cong \mathcal{L}(D_2)$$

$CH^1(X) = \text{div}/\text{rat. eq} =$ isom classes of line bundles $= \text{Pic}(X)$

$= H^1(X, \mathcal{O}_X^*)$

$$\underset{\text{(GAGA)}}{\cong} H^1(X^{an}, \mathcal{O}^*_{X^{an}})$$ X^{an} = assoc. complex analytic variety

Exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \underset{\substack{\| \\ CH^1(X)}}{H^1(X,\mathcal{O}_X^*)} \xrightarrow{c_1} H^2(X,\mathbb{Z}).$$

Facts: (i) $c_1 = cl$.

$\Rightarrow$

(ii) $CH^1(X)_0 \xrightarrow{\sim} H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ (connected component of Picard variety)

$$\begin{array}{ccccc} \mathrm{Pic}^0(X) = CH^1(X)_0 & \xrightarrow{\sim} & \dfrac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})} = \dfrac{H^{0,1}(X)}{H^1(X,\mathbb{Z})} & & \\ \| & & \downarrow \wr\ \text{P.D.} & & \\ \dfrac{Z^1(X)_0}{Z^1(X)_{rat}} & \overset{AJ}{\dashrightarrow} & \dfrac{H^{n,n-1}(X)^\vee}{H_{2n-1}(X,\mathbb{Z})} & & \end{array}$$

= complex torus (in fact it is an abelian variety)

$Z \in Z^1(X)_0$

$\Rightarrow Z = \partial\varphi$, φ has real dim $2n-1$ $\longmapsto$ $\left(\omega \mapsto \int_\varphi \omega\right)$

(iii) Analogue of AJ theorem: This diagram commutes.

(iv) On $X \times \mathrm{Pic}^0(X)$, there is a "Poincare bundle", $\mathcal{P}$, such that $\forall\ t \in \mathrm{Pic}^0(X)$,

$$\mathcal{P}_{(t)} \xrightarrow{\sim} \mathcal{L}(t).$$

$$\Rightarrow Z^1(X)_{alg} = Z^1(X)_0$$

$$\underbrace{Z^1(X)_{rat} \subseteq Z^1(X)_{alg}}_{\text{Picard variety}} = \underbrace{Z^1(X)_0 \subseteq Z^1(X)}_{\hookrightarrow H^2(X,\mathbb{Z})}$$

This gives a very satisfactory description in the case of divisors.

Arbitrary n and k now:

Analogue of Jacobian = Griffiths intermediate Jacobian

$$Z^k(X)_0 \xrightarrow{AJ} J^k(X)$$

$$H^{2n-2k+1}(X,\mathbb{C}) = \mathrm{Fil}^{n-k+1} \oplus \overline{\mathrm{Fil}^{n-k+1}}$$

$$\mathrm{Fil}^{n-k+1}(X,\mathbb{C}) = \bigoplus_{i \geq n-k+1} H^{i,2n-2k+1-i}(X,\mathbb{C})$$

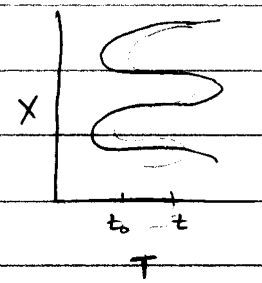
$$J^k(X) = \frac{\mathrm{Fil}^{n-k+1}(X,\mathbb{C})^\vee}{\mathrm{Im}\, H_{2n-2k+1}(X,\mathbb{Z})} = \text{a complex torus}$$

Example: $n=3 \quad k=2$

$$J^2(X) = \frac{(H^{2,1}(X) \oplus H^{3,0}(X))^\vee}{\mathrm{Im}\, H_3(X,\mathbb{Z})} = \text{a complex torus}$$

AJ is defined by

$$Z = \partial Y \longmapsto \left(\omega \mapsto \int_Y \omega\right).$$



$Z \in Z^k(X \times T)$

$t \longmapsto AJ(Z(t) - Z(t_0)) \in J^k(X)$

$T \longrightarrow J^k(X)$ is holomorphic.

$\Rightarrow$ AJ factors through

$$CH^k(X)_0 \xrightarrow{AJ} J^k(X).$$

Remarks: (i) $J^k(X)$ is not in general an abelian variety

(ii) AJ is not injective, even in the case of points on surfaces. (Mumford: ker is "very large")

Def: $J_a^k(X) :=$ largest subtorus of $J^k(X)$ whose tangent space is contained in $H^{n-k+1,n-k}(X)^\vee$.

In fact, $AJ(CH^k(X)_{alg}) \subseteq J_a^k(X)$.

(iii) $Z^k(X)_0 / Z^k(X)_{alg}$ is countable (follows from theory of Chow varieties / Hilbert schemes)

So if $Fil^{n-k+1} H^{2n-2k+1}(X) \supsetneq H^{n-k+1,n-k}(X)$, then AJ cannot be surjective.

(iv) $J_a^k(X)$ is an abelian variety. The Hodge conjecture