Kakde

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Noncommutative main conjecture of Iwasawa theory for totally real # field:

$$\begin{array}{lll}
M & & \text{max. ab. } p\text{-ext of } F_\infty \text{ unram. outside } \Sigma \\
\mid X & & \\
F_\infty & & \text{totally real Galois ext of } F \\
\mid H & & \\
F^{cyc} & \mathcal{G} & \qquad p \text{ odd prime} \\
\mid \mathbb{Z}_p \simeq \Gamma & & \\
F & & \text{totally real \# field.}
\end{array}$$

We assume that $\mathcal{G}$ is a p-adic Lie group and only finitely primes in F ramify in F_∞.
Σ - finite set of finite primes of F containing all that ramify in F_∞.

$$1 \longrightarrow X \longrightarrow Gal(M/F) \longrightarrow \mathcal{G} \longrightarrow 1$$

$\mathcal{G}$ acts on X since X is pro-p.
X is a $\mathbb{Z}_p[\mathcal{G}]$-module.
Since X is compact the action extends to an action of $\mathbb{Z}_p[[\mathcal{G}]] = \varprojlim_{U \trianglelefteq_{open} \mathcal{G}} \mathbb{Z}_p[\mathcal{G}/U] =: \Lambda(\mathcal{G})$.

Aim: To understand X as a $\Lambda(\mathcal{G})$-module.

Classical situation:

Assume that $\mathcal{G}$ is abelian and H is finite. (This is no assumption if Leopoldt's conjecture holds for F, p).

$\mathcal{G} \simeq H \times \Gamma$. Fix a top. gen. γ of Γ.

$\Lambda(\mathcal{G}) \cong \mathbb{Z}_p[H][[T]]$.

Deligne-Ribet, Cassou-Noguès, Barsky: There exists $\zeta = \zeta(F_\infty/F)$ in $\mathrm{Frac}(\Lambda(\mathcal{G}))$ s.t. for any character χ of H

$$\mathbb{Z}_p[H][[T]] \xrightarrow{\chi} \mathcal{O}_\chi[[T]] \qquad \mathcal{O}_\chi = \mathbb{Z}_p(\text{values of } \chi)$$

$$\chi(\zeta) = \begin{cases} G_\chi(T) & \text{if } \chi \neq 1 \\ \dfrac{G_\chi(T)}{T} & \text{if } \chi = 1 \end{cases}$$

p-adic cyclotomic char: $\eta_F : \mathrm{Gal}(F(\mu_{p^\infty})/F) \longrightarrow \mathbb{Z}_p^\times$

$$\eta_F(\gamma) = u.$$

Then $G_\chi(T)$ satisfies for any positive integer r divisible by $p-1$,

$$G_\chi(u^s - 1) = L_\Sigma(\chi, 1-s)$$

$$\frac{G_1(u^s-1)}{u^s - 1} = L_\Sigma(1, 1-s).$$

$V = (X \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p)$ f.d. $\mathbb{Q}_p$ v.s.

$$V = \bigoplus_{\chi \in \hat{H}} V^{(\chi)}$$

Let $f_\chi(T)$ be the characteristic poly. of $(\gamma - 1)$ acting on $V^{(\chi)}$.

Write $G_\chi(T) = \pi_\chi^{\mu_\chi} G_\chi^*(T)$ where π_χ is a unif. of $\mathcal{O}_\chi$ and

$\pi_\chi X \; G^*_\chi(T)$.

Classical main conjecture (Wiles Thm):

$$f_\chi(T)\,\mathcal{O}_\chi[[T]] = G^*_\chi(T)\,\mathcal{O}_\chi[[T]]$$

$\forall\ \chi \in \hat{H}$.

Problem in formulating H-equivariant main conjecture is that f.g module over $\Lambda(\mathcal{G})$ may not have a structure theory.

$\mathcal{N}(\mathcal{G})$ = category of perfect complexes of $\Lambda(\mathcal{G})$-modules with torsion cohomologies.

perfect complexes: complexes of $\Lambda(\mathcal{G})$-modules which are quasi-isomorphic to bounded complexes of f.g. proj. $\Lambda(\mathcal{G})$-modules.

torsion cohomologies: $\mathrm{Frac}(\Lambda(\mathcal{G})) \otimes_{\Lambda(\mathcal{G})} C^\bullet$ is acyclic.

X is $\Lambda(\mathcal{G})$-torsion.

$C = C^\bullet(F_\infty/F) = R\Gamma_c(\mathrm{Gal}(F_\Sigma/F), \mathbb{Z}_p(1))$

F_Σ is max. ext. of F unramified outside Σ.

$$H^i(C^\bullet) = \begin{cases} X & \text{if } i=2 \\ \mathbb{Z}_p & \text{if } i=3 \\ 0 & \text{o/w} \end{cases}$$

is a perfect complex.

$K_0(\mathfrak{N}(\mathcal{G}))$ free abelian group

Generators : $[D]$

Relations : ① $[D_1] = [D_2]$ if $D_1 \simeq D_2$

② If $0 \to D_1 \to D_2 \to D_3 \to 0$

then $[D_2] = [D_1] + [D_3]$.

$$\Lambda(\mathcal{G})^\times \longrightarrow \mathrm{Frac}(\Lambda(\mathcal{G}))^\times \xrightarrow{\partial} K_0(\mathfrak{N}(\mathcal{G})) \longrightarrow 0$$

$$f = \tfrac{a}{b} \longmapsto [\Lambda(\mathcal{G}) \xrightarrow{a} \Lambda(\mathcal{G})] - [\Lambda(\mathcal{G}) \xrightarrow{b} \Lambda(\mathcal{G})].$$

Example: If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda(\mathcal{G}) = \mathbb{Z}_p[[T]]$

$$R = \bigoplus_{i=1}^{n} \mathbb{Z}_p[[T]]/(f_i)$$

R is a torsion $\Lambda(\mathcal{G})$-module.

$$\mathbb{Z}_p[[T]]^n \xrightarrow{(f_1 \ldots f_n)} \mathbb{Z}_p[[T]]^n \longrightarrow R \to 0$$

$[R] \in K_0(\mathfrak{N}(\mathcal{G}))$

Then

$$\partial\left(\prod f_i\right) = [R]$$

characteristic elt. of R.

Def: Characteristic element of a class $[D]$ in $K_0(\mathfrak{N}(\mathcal{G}))$ is any element of $\mathrm{Frac}(\Lambda(\mathcal{G}))^\times$ which maps to $[D]$ under ∂.

(C.[redacted].K.S.V)

Main conjecture: (Coates, ...)

$$\partial(\zeta(F_\infty/F)) = [C(F_\infty/F)].$$

Remark: At present we can deduce this from Wiles' Theorem only is that the cyclotomic primes of F_∞^Γ is 0.

That summarizes the situation in the abelian setting.

Now let $\mathcal{G}$ be an arbitrary p-adic Lie group.

$$S = \{ f \in \Lambda(\mathcal{G}) : \Lambda(\mathcal{G})/\Lambda(\mathcal{G})f \text{ is a f.g. } \Lambda(H)\text{-module} \}.$$

Fact (c.[illegible].k.s.v): S is multiplicatively closed, does not contain any zero divisors, ~~S is an~~ Ore set, i.e., for any $s \in S$ and $r \in \Lambda(\mathcal{G})$ $\exists\, t_1, t_2 \in S$ and $w_1, w_2 \in \Lambda(\mathcal{G})$ s.t. $s w_1 = r t_1$, $w_2 s = t_2 r$.

We localize at S to get

$$\Lambda(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G})_S .$$

$$\Lambda(\mathcal{G})^\times \longrightarrow \Lambda(\mathcal{G})_S^\times \longrightarrow K_0(\mathfrak{M}_H(\mathcal{G})) \to 0$$

$\mathfrak{M}_H(\mathcal{G})$ is the category of $\Lambda(\mathcal{G})$-modules with S torsion cohomologies.

Definition of K_1 of a ring:

Let Λ be a ring with 1.

$$GL_n(\Lambda) \hookrightarrow GL_{n+1}(\Lambda)$$
$$A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

$$GL(\Lambda) = \cup GL_n(\Lambda)$$

$$K_1(\Lambda) = GL(\Lambda)/[GL(\Lambda), GL(\Lambda)]$$

Example: If Λ is commutative then $K_1(\Lambda) \xrightarrow{\det} \Lambda^\times$.
If Λ is a field or a semi-local, or a ring of integers in a # field then

$$K_1(\Lambda) \simeq \Lambda^\times.$$

Localization sequence in K-theory:

$$K_1(\Lambda(\mathcal{G})) \longrightarrow K_1(\Lambda(\mathcal{G})_S) \xrightarrow{\partial} K_0(\mathfrak{M}_H(\mathcal{G})) \to 0$$

Def: If G is a finite group then define

$$SK_1(\mathbb{Z}_p[G]) = \ker(K_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{Q}_p[G])).$$

$$SK_1(\Lambda(\mathcal{G})) = \varprojlim_{U \trianglelefteq_{\text{open}} \mathcal{G}} SK_1(\mathbb{Z}_p[G/U])$$

$$K_1'(\Lambda(\mathcal{G})) = K_1(\Lambda(\mathcal{G}))/SK_1(\Lambda(\mathcal{G})).$$

$$K_1'(\Lambda(\mathcal{G})_S) = K_1(\Lambda(\mathcal{G})_S)/\mathrm{Im}(SK_1(\Lambda(\mathcal{G}))).$$

$$K_1'(\Lambda(\mathcal{G})) \to K_1'(\Lambda(\mathcal{G})_S) \xrightarrow{\partial} K_0(\mathfrak{M}_H(\mathcal{G})) \to 0$$

Assumption: ($\mu = 0$) $C(F_\infty/F)$ is S-torsion.

$L/\mathbb{Q}_p$ finite ext. $\mathcal{O} = \mathcal{O}_L$

Let $\rho : \mathcal{G} \longrightarrow GL_n(\mathcal{O})$ cont. homom.

$\rho : \Lambda(\mathcal{G}) \longrightarrow M_n(\mathcal{O})$

$$\Phi_\rho : \Lambda(\mathcal{G}) \longrightarrow M_n(\mathcal{O}) \otimes \Lambda(\Gamma) = M_n(\Lambda_{\mathcal{O}}(\Gamma)).$$
$$g \longmapsto \rho(g) \otimes \bar{g}$$

Fact (C.F.K.S.V.) Extend to a map

$$\Phi_\rho : \Lambda(\mathcal{G})_S \longrightarrow M_n(\mathrm{Frac}(\Lambda_{\mathcal{O}}(\Gamma))).$$

$$\begin{tikzcd}
K_1(\Lambda(\mathcal{G})_S) \arrow[r] \arrow[ddddr, "\xi \mapsto \xi(\rho)"'] & K_1(M_n(\mathrm{Frac}(\Lambda_{\mathcal{O}}(\Gamma)))) \\
& K_1(\mathrm{Frac}(\Lambda_{\mathcal{O}}(\Gamma))) \arrow[u, "\cong"'] \\
& \mathrm{Frac}(\Lambda_{\mathcal{O}}(\Gamma))^\times \arrow[u, equal] \arrow[d] \\
& L \cup \{\infty\}
\end{tikzcd}$$

$$\Lambda_{\mathcal{O}}(\Gamma) \xrightarrow{\text{aug.}} \mathcal{O}$$

Main Conjecture: $\exists!\ \xi = \xi(F_\infty/F)$ in $K_1'(\Lambda(\mathcal{G})_S)$ s.t.

① $\partial(\xi) = [C(F_\infty/F)]$

② $\forall$ Artin rep ρ of $\mathcal{G}$ and any positive integer r divisible by $p-1$

$$\xi(\rho\kappa_F^{r}) = L_\Sigma(\rho, 1-r).$$