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6-17-10
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$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \bar{\mathbb{Q}}^\times$

$E = \mathbb{Q}_p(\chi)$

<u>Goal:</u> Construct $[\kappa] \in H^1(G_{\mathbb{Q}}, E(\chi^{-1}))$

(s.t. $[\kappa]$ unramified at $\ell \neq p$, automatic since $H^1(I_\ell, E(\chi^{-1})) = 0$)

s.t. $[\kappa]_p = \mathcal{L}_{an}(\chi)\, \mathrm{ord}_p + \log_p$ in $H^1(G_p, E) = \mathrm{Hom}_{cts}(\mathbb{Q}_p^\times, E)$.

$$E_1(1,\chi) = \frac{1}{2}L(\chi,0) + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi(d)\Big)q^n$$

$$\mathcal{E}_k(1,\chi) = \frac{1}{2}L_p(\chi\omega, 1-k) + \sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ (d,p)=1}}\chi(d)\langle d\rangle^{k-1}\Big)q^n$$

<u>Note:</u> $\mathcal{E}_1(1,\chi) = E_1(1,\chi\omega^0) = E_1(1,\chi)(z) - E_1(1,\chi)(pz)$

$$\mathcal{G}_k = \frac{\mathcal{E}_k(1,\omega^{-1})}{\text{its constant term}} = 1 + \frac{2}{\zeta_p(1-k)}\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n \\ (d,p)=1}}\omega^{-1}(d)\langle d\rangle^{k-1}\Big)q^n$$

$\mathcal{H}_k = \mathcal{E}_k(1,\chi) - E_1(1,\chi)\,\mathcal{G}_{k-1}\cdot\dfrac{L_p(\chi\omega,1-k)}{L(\chi,0)}$. This has constant term 0.

$\mathcal{H}_1 = \mathcal{E}_1(1,\chi)$ So $\mathcal{H}_k$ looks like a cuspform for wt $k \geq 2$, but at $k=1$ is Eisenstein.

$$\mathcal{H}_k = \sum_{\substack{(\eta,\psi) \\ \eta\psi=\chi}} c_k(\eta,\psi)\,\mathcal{E}_k(\eta,\psi) + \text{cusp form} \quad \forall\ k \geq 2.$$

Since $\mathcal{H}_k$ has constant term 0, $c_k(1,\chi) = 0$.

By slashing w/ $\begin{pmatrix} 1 & x \\ p & N \end{pmatrix} \in SL_2(\mathbb{Z})$ and then compare constant terms, we obtain

$$c_k(\chi,1) = -\frac{L(\chi^{-1},0)}{L_p(\chi^{-1}\omega,1-k)}\cdot\frac{L_p(\chi\omega,1-k)}{L(\chi,0)}\langle N\rangle^{k-1}$$

Let $\mathcal{F} = \mathcal{H} - c(\chi,1)\mathcal{E}(\chi,1)$. This is "closer" to being a cusp form.

$\mathcal{F}$ is a "mod $(k-1)^2$ - eigenform."

Let $I = \underset{\substack{\uparrow \\ \text{specialization to wt 1} \\ \text{map}}}{\mathrm{Ker}(\nu_1)} = (u(1+T)-1)$. Let Λ_I = localization of Λ at I.

$$\Lambda_I/I^2 \simeq E[x]/x^2$$

$$\lambda \longmapsto \lambda(1) + \lambda'(1)x \qquad \left(\begin{array}{l}\text{viewing } \lambda \text{ as a} \\ \text{function of } \kappa \in \mathbb{Z}_p\end{array}\right)$$

<u>Prop:</u> $\exists$ constants $\lambda_\ell \in \Lambda_I/I^2$ s.t.

$$\left.\begin{array}{ll} T_\ell \mathcal{F} \equiv \lambda_\ell \mathcal{F} & \\ U_\ell \mathcal{F} \equiv \lambda_\ell \mathcal{F} & \end{array}\right. \pmod{I^2} \quad \begin{array}{l} \ell \nmid Np \\ \ell \mid Np. \end{array}$$

where

$$\lambda_\ell = (1+\chi(\ell)) + x \cdot \log_p(\ell)(\alpha + \chi(\ell)\beta), \quad \ell \neq p$$

$$\lambda_p = 1 + x\beta\mathcal{L}_{an}(\chi)$$

with

$$\alpha = \frac{\mathcal{L}_{an}(\chi)}{\mathcal{L}_{an}(\chi)+\mathcal{L}_{an}(\chi^{-1})} \qquad \beta = 1-\alpha.$$

<u>Assumption:</u> $\mathcal{L}_{an}(\chi) + \mathcal{L}_{an}(\chi^{-1}) \neq 0$.

We can arrange $\beta \neq 0$ by possibly switching χ and χ^{-1}.

To prove this, one just does the explicit calculation. Let $m = bp^t$ with $p \nmid b$, then we have

$$a_m(\mathcal{E}(1,\chi)) \equiv \sum_{d|b} \chi(d)(1+\log_p(d)\cdot x)$$

(mod I^2)

$$a_m(\mathcal{E}(\chi,1)) \equiv \sum_{d|b} \chi(d)(1+\log_p(\tfrac{b}{d})x).$$

$$\mathcal{C}_{k-1}\frac{L_p(\chi\omega,1-k)}{L(\chi,0)} \equiv -\mathcal{L}_{an}(\chi)\,x \pmod{I^2}$$

Combining all of this gives:

$$a_m(\mathcal{F}) \equiv \left(\sum_{d|b}\Psi_1(\tfrac{b}{d})\Psi_2(d)\right)(1+\beta\mathcal{L}_{an}(\chi)\,x)^t \pmod{I^2}$$

where Ψ_1 and Ψ_2 are totally mult. functions on $\mathbb{Z}_{\geq 1}$ defined by

$$\Psi_1(l) = 1+\alpha\log_p(l)\cdot x$$

$$\Psi_2(l) = \chi(l)(1+\beta\cdot\log_p(l)\cdot x).$$

It then follows (as for Eisenstein series) that $\mathcal{F}$ has Hecke eigenvalues

$\Psi_1(l)+\Psi_2(l)$ for $l \neq p$

$1+\beta\mathcal{L}_{an}\cdot x$ for $l = p$.

$$^{\sigma}\mathcal{F} = \sum_{\substack{\eta,\psi \\ \eta\psi=\chi \\ (\eta,\psi)\neq(1,\chi) \\ \text{or }(\chi,1)}} c(\eta,\psi)\,\mathcal{E}(\eta,\psi) + \text{cusp form.}$$

For each $(\eta,\psi) \neq \begin{cases}(1,\chi)\\(\chi,1)\end{cases}$, chose l s.t. $\eta(l)+\chi(l) \neq 1+\chi(l)$.

Consider the Hecke operator

$$t = \eta(\ell) + \psi(\ell)\langle \ell \rangle^{k-1} - T_\ell$$

This annihilates $\mathcal{E}(\eta, \psi)$ and on $\mathcal{F}$ has a mod I^2 eigenvalue that is invertible, ~~so~~ replace $\mathcal{F}$ by $\frac{t(\mathcal{F})}{a_1(t\mathcal{F})}$, get a Λ_I-adic form with the same mod I^2 eigenvalues as $\mathcal{F}$. We do this with each pair until we have Λ_I-adic cusp form with

$$\mathcal{F}_1 = \mathcal{E}_1(1, \chi) = E_1(1, \chi\omega^0),$$

that is a mod I^2 eigenform with eigenvalues as in the previous prop.

Apply Hida's e to make it ordinary.

<u>Note:</u> ψ_1 and ψ_2 can be viewed as Galois characters:

$$\psi_1, \psi_2 : G_{\mathbb{Q}} \longrightarrow \left(E[x]/x^2\right)^\times$$

$$\psi_1(\sigma) = 1 + \alpha \log_p \varepsilon(\sigma)\, x$$

$$\psi_2(\sigma) = \chi(\sigma)(1 + \beta \log_p \varepsilon(\sigma) x).$$

$\mathbb{T}$ = Hida's Hecke algebra of $\mathcal{S}(N, \chi)^0$.

$$\mathfrak{P} = \ker(\mathbb{T} \longrightarrow E)$$

$$T_\ell \longmapsto 1 + \chi(\ell) \qquad \ell \nmid Np$$

$$U_\ell \longmapsto 1 \qquad \ell \mid Np$$

corresponding to $\mathcal{E}_1(1, \chi)$.

$\mathcal{F}$ gives rise to a map $\phi : \mathbb{T}_{\mathfrak{P}} \longrightarrow E[x]/x^2 \simeq \Lambda_I/I^2$.

given by

$$\begin{array}{ll} \ell \nmid N: & T_\ell \\ \ell \mid N: & U_\ell \end{array} \longmapsto \psi_1(\ell) + \psi_2(\ell)$$

$$U_p \longmapsto 1 + \beta\, \mathcal{L}_{an}(\chi)\, X .$$

$$\mathbb{T}_{\mathfrak{P}} \longleftrightarrow L_1 \times \cdots \times L_n \qquad L_i = \mathrm{Frac}(\Lambda_i)$$

$$\Lambda_i = \Lambda\left[\begin{array}{c}\text{Hecke eigenvalues}\\ \text{of } D_i\end{array}\right]_I$$

$$T_\ell \longmapsto (a_\ell(D_1), \ldots, a_\ell(D_n))$$

(a finite ext. of Λ_I)

$D_i = \Lambda$-adic cuspidal e.f.'s s.t.

$$\nu_1(D_i) = \mathcal{E}_1(1, \chi)$$

(Question by Hida as to how we know $\mathbb{T}_\mathfrak{P}$ is reduced.. could get nilpotents from oldforms.. Choose $N = \mathrm{cond}(\chi)$ and then it should be ok..)

$$\rho: G_\mathbb{Q} \longrightarrow GL_2(L_1 \times \cdots \times L_n)$$

- unramified outside Np.
- $\forall\, \ell \nmid Np$, $\mathrm{tr}(\rho(\mathrm{Frob}_\ell)) = T_\ell$

$$\det(\rho(\mathrm{Frob}_\ell)) = \chi(\ell)\langle \ell \rangle^{k-1} \swarrow$$

elt of Λ that represents the fctn $\ell \mapsto \ell^{k-1}$

δ = complex conj.

Choose a basis s.t.

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Write $\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$.

Prop: ① $a(\sigma), d(\sigma) \in \mathbb{T}_\beta$

$\phi \circ a = \psi_1$, $\phi \circ d = \psi_2$

② The projection of $b(\sigma)$ to any L_i is not the zero function when restricted to G_p.

Pf: ① $Tr(\rho(Frob_\ell)) = T_\ell \in \mathbb{T}$. Cheb. $\Rightarrow$ $Tr(\rho(\sigma)) \in \mathbb{T}$ for all $\sigma \in G_{\mathbb{Q}}$.

$$a(\sigma) = \frac{1}{2}(Tr(\rho(\sigma)) + Tr(\rho(\sigma\delta))) \in \mathbb{T}$$

$$d(\sigma) = \frac{1}{2}(Tr(\rho(\sigma)) - Tr(\rho(\sigma\delta))) \in \mathbb{T}$$

$\phi(Tr(\rho\sigma)) = \psi_1(\sigma) + \psi_2(\sigma)$ combined with and $\psi_1(\delta) = 1$, $\psi_2(\delta) = -1$ gives the rest of part ①.

② Let $B = \Lambda_i$ - submodule of L_i generated by the projection of $b(\sigma)$ $\forall$ $\sigma \in G_{\mathbb{Q}}$. B f.g Λ_I -module by compactness.

Let $\mathfrak{m}$ = maximal ideal of Λ_i.

$proj_{L_i} d(\sigma) \bmod \mathfrak{m} \equiv \psi_2(\sigma) \bmod I \equiv \chi(\sigma)$

$\therefore$ $proj_{L_i} d(\sigma)$ is invertible in Λ_i.

Let $\kappa(\sigma) = proj_{L_i}\left(\frac{b(\sigma)}{d(\sigma)}\right) \in B$.

$\bar{\kappa} : G_{\mathbb{Q}} \longrightarrow \bar{B} = B/\mathfrak{m}B$ is a cts. 1-cocycle in $Z^1(G_{\mathbb{Q}}, \bar{B}(\chi^{-1}))$.

If $b(\sigma) = 0$ $\forall$ $\sigma \in G_p$, then $[\bar{\kappa}]_p$ is 0, hence $[\bar{\kappa}] = 0$. If so, we can write

$$\bar{\kappa}(\sigma) = (\chi^{-1}(\sigma) - 1)y \quad \forall \sigma \in G_{\mathbb{Q}} \quad \text{for some } y.$$

Plug in $\sigma = \delta$ = complex conj.

$$0 = \bar{\kappa}(\delta) = -2y \Rightarrow y = 0 \Rightarrow \bar{\kappa} = 0 \text{ on } G_{\mathbb{Q}}.$$

$\Rightarrow$ by Nakayama $\kappa(\sigma)=0$ $\forall\sigma$ $\Rightarrow \mathrm{proj}_{L_i}\, b(\sigma)=0$ $\forall\sigma$

This is a contradiction b/c the rep. is irreducible for each cuspidal e.f. ■

Since the forms D_i are ordinary, ∃ a change of basis matrix.

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\in GL_2(L_1\times\cdots\times L_n)\quad s.t.$$

$$\begin{pmatrix}a(\sigma)&b(\sigma)\\c(\sigma)&d(\sigma)\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}\varepsilon^{k-1}\eta^{-1}(\sigma)&*\\0&\eta(\sigma)\end{pmatrix}\quad\forall\sigma\in G_p.$$

$\eta: G_p \longrightarrow \mathbb{T}_{\mathfrak{P}}^{\times}$ unram,

$\eta(\mathrm{Frob}_p)=U_p.$

Upper left entry

$$C\cdot b(\sigma)=A\left[\varepsilon^{k-1}\eta^{-1}(\sigma)-a(\sigma)\right].$$

The second part of the prop. implies the projection of A onto any L_i is not zero.

Let $\tilde{b}(\sigma)= b(\sigma)\cdot\frac{C}{A}\in L_1\times\cdots\times L_n.$

Let $B=\mathbb{T}_{\mathfrak{P}}$ - submodule generated by the $\tilde{b}(\sigma)$ $\forall\sigma\in G_{\mathbb{Q}}$.

$$K(\sigma)=\frac{\tilde{b}(\sigma)}{d(\sigma)}$$

$$(*)\quad K(\sigma)=\frac{\varepsilon^{k-1}\eta^{-1}(\sigma)-a(\sigma)}{d(\sigma)}\quad\forall\sigma\in G_p$$

Claim: $B\subset\mathfrak{P}\subset\mathbb{T}_{\mathfrak{P}}$

Pf sketch: Mimic the proof of part ③ of the prop. with the

image of κ in $\bar{B}^{\#} = B^{\#}/\mathcal{B}B^{\#}$ where $B^{\#} = (B+\mathcal{B})/\mathcal{B}$.
You then show $B^{\#} = 0$, which is what is wanted. $\square$

κ takes values in $\mathcal{B}$. $\phi(\kappa(\sigma)) = \kappa(\sigma)\cdot X$ in $E[X]/X^2$.
This defines $\kappa(\sigma)$.
Then

$$[\kappa] \in H^1(G_{\mathbb{Q}}, E(\chi^{-1})).$$

$$\text{Using}\quad \left.\begin{array}{l} \varepsilon^{\kappa-1}(\sigma) \equiv 1 + \log_p \varepsilon(\sigma)\cdot X \\ \phi\circ\eta\,(\mathrm{Frob}_p) = 1 + \beta\,\mathcal{L}_{an}(\chi)\cdot X \\ \phi\circ a(\sigma) = \psi_1(\sigma) \\ \phi\circ d(\sigma) = \psi_2(\sigma) \end{array}\right\} \Rightarrow \quad [\kappa]_p = \beta\left(\mathcal{L}_{an}(\chi)\,\mathrm{ord}_p + \log_p\right).$$