Diagonalizing Spaces of Siegel Eisenstein Series

Walling

$Sp_n(\mathbb{Z})$ $\quad 2n \times 2n$

Fix $k, n, N \in \mathbb{Z}_+$, χ char mod N

$\Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z}) : C \equiv 0 \ (\mathrm{mod}\, N) \right\}$

$\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in Sp_n(\mathbb{Z}) \right\}$

For $\gamma \in Sp_n(\mathbb{Z})$, set $\mathbb{E}_\gamma(\tau) = \sum_{\delta} \overline{\chi}(\delta)\, 1(\tau)|\gamma\delta$ where

$\Gamma_\infty \gamma \Gamma_0(N) = \coprod_\delta \Gamma_\infty \gamma \delta$, $\quad 1(\tau)|\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(C\tau + D)^{-k}$.

$\chi(\delta) = \chi(\det D_\delta)$.

Need $k > n+1$ for convergence.

Take a set $\{\gamma_\sigma\}$ of reps for $\Gamma_\infty \backslash Sp_n(\mathbb{Z}) / \Gamma_0(N)$. Can show that we can choose $\gamma_\sigma = \begin{pmatrix} I & 0 \\ M_\sigma & I \end{pmatrix}$. Also note $\begin{pmatrix} * & * \\ M & N \end{pmatrix} \in Sp_n(\mathbb{Z})$ iff (M, N) is a coprime symmetric pair, meaning $M\,{}^tN = N\,{}^tM$ and for all primes p, $\mathrm{rk}_p(M, N) = n$.

$\mathbb{E}_{\gamma_\sigma}(\tau) = \sum_{(M,N)} \overline{\chi}(M,N) \det(M\tau + N)^{-k}$ where $GL_n(\mathbb{Z})(M,N)$ runs over $GL_n(\mathbb{Z})(M_\sigma\, I)\,\Gamma_0(N)$.

Fix a prime $p \nmid N$. Evaluate $\mathbb{E}_{\gamma_\sigma} | T(p)$ and $\mathbb{E}_{\gamma_\sigma} | T_j(p^2)$.

Let $\Gamma = \Gamma_0(N)$ and $\delta = \begin{pmatrix} pI_n & 0 \\ 0 & I_n \end{pmatrix}$. Let $\{\beta\}$ be a set of reps for $\delta\Gamma\delta^{-1} \cap \Gamma \backslash \Gamma$. Then

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$$f \mid T(p) = p^{n(k-n-1)/2} \sum_{\beta} \bar{\chi}(\beta)\, f \mid \delta^{-1}\beta.$$

For $0 \le r \le n$, set $X_r = \begin{pmatrix} p\, I_r & \\ & I_{n-r} \end{pmatrix}$,

$K_r = X_r GL_n(\mathbb{Z}) X_r^{-1} \cap GL_n(\mathbb{Z})$. Then for each r, let G vary over $GL_n(\mathbb{Z})/K_r$, $y = \begin{pmatrix} y_0 & 0 \\ 0 & 0 \end{pmatrix}$, $y_0 \in \mathbb{Z}^{r,r}_{sym}$ varying mod p.

$$\mathbb{E}_{\gamma_r}(\tau) \mid T(p) = p^* \sum_{r=0}^{n} \sum_{\substack{(M,N) \\ G,y}} \bar{\chi}(M,N)\, \chi(p^{n-r}) \det\left(M(X_r^{-1}G^{-1}\tau + X_r^{-1} y\, {}^tG)\, p\, {}^tG^{-1}X_r^{-1} + N\right)^{-k}$$

$$= p^* \sum_{r=0}^{n} \sum_{\substack{(M,N) \\ G,y^*}} \bar{\chi}(M,N)\, \chi(p^{n-r}) \det\left(pMX_r^{-1}G^{-1}\tau + pMX_r^{-1}y\,{}^tG + N X_r\, {}^tG\right)^{-k}$$

<u>Note:</u> For any σ', $\beta' = \begin{pmatrix} I & 0 \\ -M_{\sigma'} & I \end{pmatrix}\begin{pmatrix} I & 0 \\ (p\bar{p})^3 M_{\sigma'} & I \end{pmatrix} \in \Gamma(N)$ where $\bar{p}p \equiv 1 \pmod{N}$.

So $\mathbb{E}_{\gamma_{\sigma'}} = \mathbb{E}_{\gamma_{\sigma'}\beta}$ $\therefore$ we can assume $p^3 \mid M_{\sigma'}$ $\forall \sigma'$.

Find s with $(M'G \;\; N'\,{}^tG^{-1}) = X_{n-s}^{-1}\,(pMX_r^{-1} \,.\, pMX_r^{-1}y + NX_r)$ s.t. M' and N' are integral and coprime. (Check : $\operatorname{rank}_p(M',N') = n$.)

Then we get $E_{\gamma_r}(\tau) \mid T(p)$ as a (weighted) sum of terms of the form $\det(M'\tau+N')^{-k}$.

We know $\mathbb{E}_{\gamma_\sigma} \mid T(p) = \sum_{\sigma'} c_{\sigma,\sigma'} \mathbb{E}_{\gamma_{\sigma'}}$.

To compute $c_{\sigma,\sigma'}$, we need to count how often $GL_n(\mathbb{Z})(M_{\sigma'}, I)$ gets hit. Equivalently, want all r, s, G, y, $GL_n(\mathbb{Z})(M,N) \in GL_n(\mathbb{Z})(M_\sigma, I)\Gamma_0(N)$ where $(M,N) = X_{n-s} E\left(\frac{1}{p} M_{\sigma'} G X_r \left({}^tG^{-1} - \frac{1}{p} M_{\sigma'} G y\right) X_r^{-1}\right)$ for some $E \in GL_n(\mathbb{Z})$. We have $GL_n(\mathbb{Z}) X_{n-s} E = GL_n(\mathbb{Z}) X_{n-s}$ iff $E \in {}^tK_{n-s}$, so we let E varying over ${}^tK_{n-s}\backslash GL_n(\mathbb{Z})$. We have $p \mid M$; when N is integral with $\operatorname{rank}_p N = n$.

<u>Need:</u> $X_{n-s} E\, {}^tG^{-1} X_r^{-1}$ is integral with p-rank n, so need $n-s=r$.

Then $\det(X_r E\, {}^tG^{-1} X_r^{-1}) = \pm 1$, so when $X_r E\, {}^tG^{-1} X_r^{-1}$ is integral, ${}^tE\, {}^tG^{-1}$ lies in ${}^tK_r$. So given G, take $E = {}^tG$.

Then $(M,N) \equiv \left(\frac{1}{p} X_r M_{\sigma'} X_r, I\right) \pmod N$. So $(M,N) \in GL_n(\mathbb{Z})(M_\sigma, I)\Gamma_0(N)$ iff $\left(\frac{1}{p} X_r M_{\sigma'} X_r, I\right)$ is,

iff $(M_{\sigma'}, I) \in GL_n(\mathbb{Z})(p X_r^{-1} M_\sigma X_r^{-1}, I)\Gamma_0(N)$

$$= GL_n(\mathbb{Z})\left(p \begin{pmatrix} p^{-r} & \\ & I \end{pmatrix} M_\sigma \begin{pmatrix} \bar{p}^{r} & \\ & I \end{pmatrix}, I\right)\Gamma_0(N).$$

<u>Group action:</u> For $v, w \in \mathcal{U}_N = (\mathbb{Z}/N\mathbb{Z})^\times$, set

$$(v,w) \cdot M_\sigma = v \begin{pmatrix} w & \\ & I \end{pmatrix} M_\sigma \begin{pmatrix} w & \\ & I \end{pmatrix},$$

$$(v,w)\cdot\gamma_\sigma = \begin{pmatrix} I & 0 \\ (v,w)M_\sigma & I \end{pmatrix}.$$

Thm:

$$\mathbb{E}_{\gamma_\sigma} | T(p) = \sum_{r=0}^{n} \chi(p^{n-r})\, p^{k(n-r)-(n-r)(n+r+1)/2}\, \beta(n,r)\, \mathbb{E}_{(p,\bar{p}^{r})\cdot\gamma_\sigma}$$

where $$\beta(n,r) = \prod_{i=0}^{r-1} \frac{(p^{n-i}-1)}{(p^{r-i}-1)}.$$

Diagonalising:

Take ψ to be a character on $\mathcal{U}_N \times \mathcal{U}_N$. Set

$$\mathbb{E}_{\sigma,\psi} = \sum_{v,w\in\mathcal{U}_N} \overline{\psi}(v,w)\, \mathbb{E}_{(v,w)\gamma_\sigma}.$$

$\psi(v,w) = \psi_1(v)\psi_2(w)$,

Cor: $\mathbb{E}_{\sigma,\psi} | T(p) = \lambda_{\sigma,\psi}(p)\, \mathbb{E}_{\sigma,\psi}$ where

$$\lambda_{\sigma,\psi}(p) = \psi_1(p)\overline{\psi}_2(p^n) \prod_{i=1}^{n} (\psi_2\chi(p)p^{k-i}+1).$$

Pf: Assume $p^3 | M_\sigma$.

$$\mathbb{E}_{\sigma,\psi}|T(p) = p^{kn-n(n+1)/2} \sum_{v,w,r} \overline{\psi}(v,w)\,\chi(p^{n-r})\, p^{-kr+r(r+1)/2}$$

$$\cdot\,\beta(n,r)\,\mathbb{E}_{(pv,\,\bar{p}^{r}w)\cdot\gamma_\sigma}$$

Change vars. $v \mapsto \bar{p} v$, $w \mapsto p^{r} w$. ~~Get~~

$$\psi_1(p)\chi(p^n)p^{*}\, S(n,k)\, E_{\gamma,\psi} \quad \text{where}$$

$$S(n,k) = \sum_{r=0}^{n} \psi_2\chi(\bar{p}^{r})\, p^{-kr + r(r+1)/2}\, \beta(n,r).$$

$$\beta(n,r) = p^{r}\,\beta(n-1,r) + \beta(n-1,r-1)$$

$$\rightsquigarrow \quad S(n,k) = (\psi_2\chi(\bar{p}^{r})\, p^{1-k} + 1)\, S(n-1,k-1).$$

Can do all this for $T_j(p^2)$; just messier.

Same process will work for half-integral weight Eisenstein series

as long as one looks at E_γ for $\gamma \in \Gamma_0(4)$.