Big Image of Galois Representations and Congruence Ideals

Tilouine

joint work w/ Hida

$k \geq 2$ $\quad \Gamma \leq SL_2(\mathbb{Z})$ level N

$S_k = S_k^{CM} \oplus S_k^{gen}$

f general $\overset{\text{Ribet}}{\Longrightarrow}$ $\rho_f : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ $\quad$ ($\bar\rho_f$ irred.)

$\exists\ \alpha \in GL_2(\mathcal{O})$, $\exists\ \underset{\neq 0}{l_f} \subseteq \mathbb{Z}_p$

$\alpha\, \Gamma_{SL_2}(l_f)\, \alpha^{-1} \subset \operatorname{Im} \rho_f$.

The best l_f is the Galois level of f at p.

If $\exists\ f \equiv g\ (p^{d})$ with g CM then $p^{b} \mid l_f$.

$S_k = \mathbb{C} f \oplus S_k'$

$$\begin{tikzcd} & \mathcal{O} \arrow[r] & \mathcal{O}/c_f \arrow[d, equal] \\ h_k \arrow[ur, "\lambda_f"] \arrow[r] & h_k' \arrow[r] & h'/c' \end{tikzcd}$$

largest ideal so that one has this is the congruence ideal.

Write $S_k = S_k^{CM} \oplus \mathbb{C}f \oplus S_k''$

$$\begin{tikzcd} & \mathcal{O} \arrow[r] & \mathcal{O}/c_{f,CM} \arrow[d, equal] \\ h_k \arrow[ur, "\lambda_f"] \arrow[r] & h_k^{CM} \arrow[r] & h^{CM}/c^{CM} \end{tikzcd}$$

$c_{f,CM} \mid c_f$

measures congruences between f and CM-forms.

We assume

$$\bar\rho_f \equiv \bar\rho_g \pmod{\mathfrak{p}} \quad \text{with } g \text{ is CM.}$$

If $\exists\ \wp | p,\ \wp \supset \mathcal{I}_{f,CM} \Rightarrow p \mid l_f$.

② Hida 2012

$\Lambda_1 = \mathbb{Z}_p[[T]]$

Γ congruence subgroup of level N, $p \nmid N$.

$h^\circ(\Gamma)$ ordinary Hecke alg.

$$\begin{tikzcd} h^\circ(\Gamma) \arrow[rr, "\lambda"] \arrow[dr, phantom, "\text{fin. torsion free}"] & & \mathbb{I} \arrow[dl, phantom, "\substack{\text{fin.}\\ \text{torsion free}\\ \text{domain}} \supset \Lambda_1"] \\ & \Lambda_1 \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd} \qquad \Lambda_1\text{- algebra homom.} \qquad \left(\begin{tikzcd} \text{analogue of} \\ h_k \arrow[r, "\lambda_f"] \arrow[dr, dash] & \mathcal{O} \arrow[d, dash] \\ & \mathbb{Z}_p \end{tikzcd} \right)$$

We assume λ is general, i.e. if we specialize to classical weights we get non-CM, i.e. ($p \geq 2$, $u = 1+p$ top. gen of $1+p\mathbb{Z}_p$.)

$\exists\ \wp \mid P_k = (1+T-u^k)$ in $\mathbb{I}$ $\rightsquigarrow \lambda_\wp : h_k(\Gamma_0(p), \mathbb{Z}_p) \longrightarrow \mathcal{O}_\wp$ is general

for one (or for all, it is equivalent)

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{I}).$$

Assume $\bar\rho_\lambda$ irred.

<u>Thm 1 (Hida)</u>: If $\exists\ \delta \in \rho_\lambda(D_p)$ with eigenvalues $\alpha, \beta \in \mathbb{I}$ s.t. $\alpha^2 \not\equiv \beta^2 \bmod \mathfrak{m}_{\mathbb{I}}$, then $\exists\ \underset{\neq 0}{l_\lambda} \subseteq \Lambda_1$,

$\alpha \in GL_2(\mathbb{I})$ s.t $\alpha\, \Gamma_{SL_2}(l_\lambda)\alpha^{-1} \subseteq \operatorname{Im} \rho_\lambda$.

Thm 2: 1) If $SL_2(\mathbb{F}_p) \subseteq \operatorname{Im} \bar\rho_\lambda$. Then l_λ is primary $(\mathfrak{m}_{\Lambda_1}^m \subseteq l_\lambda)$.

2) if $\bar\rho_\lambda = \operatorname{Ind}_{G_K}^{G_\mathbb{Q}} \bar\mu$ (mod $\tilde{\mathfrak{m}}_{\mathbb{I}}$) then with

$L_{\lambda, cm} \subseteq \mathbb{I}$ defined as before,

$$L'_{\lambda, cm} = L_{\lambda, cm} \cap \Lambda_1.$$

$\forall\ \underline{P} \in (\operatorname{Spec} \Lambda_1)_{ht=1}$, (another small technical assump)

$$\operatorname{ord}_{\underline{P}}(L'_{\lambda, cm}) \leq \operatorname{ord}_{\underline{P}}(l_\lambda) \leq 2 \operatorname{ord}_{\underline{P}}(L'_{\lambda, cm}).$$

3) Case of GSp_4.

$k_1 \geq k_2 \geq 3$. integers

$\underline{k} = (k_1, k_2)$.

$W_{\underline{k}} \circlearrowleft GL_2$

Define the space of vector-valued Siegel modular forms $S_{\underline{k}}(\Gamma)$.

$$\mathcal{H}^{N, sph} \searrow h_{\underline{k}}(\Gamma) \circlearrowleft S_{\underline{k}}(\Gamma)$$

$$\Gamma_B(p) = \Gamma \cap \{\gamma \in Sp_4(\mathbb{Z}) : \gamma \bmod p \in B(\mathbb{Z}/p\mathbb{Z})\}$$

This is contained in Siegel parabolic, paramodular, etc. This is the group needed for Hida theory.

$p \nmid N$, Γ level N.

<u>Def</u>: $f \in S_{\underline{k}}(\Gamma)$ <u>p-ordinary</u>:

1)

$$T_{p,1} \longleftrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix} \qquad a_{p,i} \quad i = 1,2$$

$$T_{p,2} \longleftrightarrow \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix} \qquad \begin{cases} \mathrm{ord}_p(a_{p,1}) = 0 \\ \mathrm{ord}_p(a_{p,2}) = k_2 - 3. \end{cases}$$

Another way to say this is for t_p = Satake parameters

$$t_p = \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix}$$ then $\mathrm{ord}_p(\alpha) = 0$, $\mathrm{ord}_p(\beta) = k_2 - 2$, $\mathrm{ord}_p(\gamma) = k_1 - 1$, $\mathrm{ord}_p(\delta) = k_1 + k_2 - 3$.

2) $f \in S_{\underline{k}}(\Gamma_B(p))$ $\quad U_p \rightsquigarrow \alpha_i$ $\quad$ p-ordinary here means

$$\begin{cases} \mathrm{ord}_p(\alpha_1) = 0 \\ \mathrm{ord}_p(\alpha_2) = 0. \end{cases}$$

3) f level $\Gamma \xrightarrow{p\text{-stab}} \tilde{f}$ level $\Gamma_B(p)$

f p-ord $\Rightarrow$ $\tilde{f}$ p-ord $\qquad$ get $\tilde{f}$ by slashing by U_p's.

$\alpha_1 = \alpha$

$\alpha_2 = \beta / p^{k_2 - 2}$.

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$$S_{\underline{k}}(\Gamma_B(p)) = \underbrace{S_{\underline{k}}(\Gamma_B(p), \mathbb{Z}[\tfrac{1}{N}]) \otimes \mathbb{C}}_{\text{stable } \mathcal{H}^{N,sph}[U_{p,i}]}.$$

Denote the Hecke alg. acting on this by $\mathfrak{h}_{\underline{k}}(\Gamma_B(p), \mathbb{Z}_p)$.

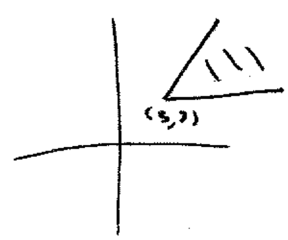
$S^{ord}_{\underline{k}}(\Gamma_B(p), \mathbb{Z}_p)$ largest subspace for which $U_{p,i}$ act as $\mathbb{Z}_p$ invertible operators. Write $\mathfrak{h}^{ord}_{\underline{k}}(\Gamma_B(p), \mathbb{Z}_p)$ is image of $\mathfrak{h}_{\underline{k}}(\Gamma_B(p), \mathbb{Z}_p)$ acting on this.

Let $\underline{i} = (i_1, i_2) \in (\mathbb{Z}/(p-1)\mathbb{Z})^2$.

$$\begin{array}{l} \mathfrak{h}^{ord}_{\underline{i}}(\Gamma) \circlearrowright S^{ord}_{\underline{i}} \circlearrowleft T(\mathbb{Z}_p) \\ \Big| \text{ fin. tor. free} \\ \Lambda_2 = \mathbb{Z}_p[[T_1, T_2]] \end{array}$$

$$T = \left\{ \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix} \right\} \subseteq Sp_4.$$

$$X(T)^+ = \underset{\underline{k}}{\mathbb{Z}^2} \cap C^+$$

C^+ 

$$S^{ord}_{\underline{i}}[T = \underline{k}] = S^{ord}_{\underline{k}}(\Gamma_B(p), \mathbb{Z}_p).$$

$\forall\ \underline{k} \in X^+ \qquad \underline{k} \equiv \underline{i} \pmod{p-1}.$

$$\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]] \curvearrowright (S_{\underline{i}}^{ord})^\vee$$

$$P_{\underline{k}} = (1 + T_1 - u^{k_1},\ 1 + T_2 - u^{k_2})$$

$$h_{\underline{i}}^{ord} / (P_{\underline{k}}) \longrightarrow\!\!\!\!\rightarrow h_{\underline{k}}^{ord}(\Gamma_B(p), \mathbb{Z}_p).$$
+ nilpot. kernel

$$\begin{tikzcd} h_{\underline{i}}^{ord} \arrow[rr] \arrow[d] & & \mathbb{I} \arrow[d] \\ & \Lambda_2 \arrow[ul, dash] \arrow[ur, dash] & \\ h_{\underline{k}}^{ord}(\Gamma_B(p), \mathbb{Z}_p) \arrow[rr, "\lambda_f"] & & \mathcal{O} \end{tikzcd}$$

$$\begin{array}{l} \mathbb{I} \\ | \ \text{fin torsion free} \\ \Lambda_2 \end{array}$$

So any form can be lifted; need extra assumptions to get it lifts uniquely.

One obtains

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GSp_4(\mathbb{I})$$

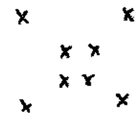
[Assume $\bar{\rho}_\lambda$ irred.]

$\underline{k} \in X^+$

$$\begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad S_{\underline{k}} = S_{\underline{k}}^{CAP,\,siegel} \oplus S_{\underline{k}}^{CAP,\,klingen} \oplus S_{\underline{k}}^{Yosh} \oplus S_{\underline{k}}^{Sym^3} \oplus S_{\underline{k}}^{Twist\ Yosh} \oplus S_{\underline{k}}^{gen}$$

Galois reps:

$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$

CAP Siegel	CAP Klingen	Yosh	
			
$k_1 = k_2$	Soudry		
SK-lift	CAP	$GL_2 \times GL_2$ w/ same det	$\forall (k_1, k_2)$
	$k_2 = 3$	$= (GL_2\times GL_2)^\circ$	

$Sym^3 GL_2$	Twist Yosh	gen
$k_1 = 2k_2 - 3$	Normalizer $(GL_2\times GL_2)^\circ$	?

for example only gives 1-var. family. So there is a 2-variable form passing through it, but that means the specializations of the 2-var. form cannot all be of this type.

We say f is <u>Galois general</u> means its Galois of representation is in ? case.

<u>Prop</u>: f Galois gen. $\Rightarrow$ ρ_f irred, $\underbrace{Ad^\circ}_{\text{on Lie alg, dim 10}} \rho_f$ irred.

(Note $Ad\,\rho_f$ is red. if f is Sym^3 or twist Yosh.)

and $\exists\ \alpha \in GSp_4(\mathcal{O})$, $\exists\ \ell_f \in \mathbb{Z}_p$, $\ell_f \neq 0$ s.t.

$\alpha\, \Gamma_{Sp_4}(\ell_f)\, \alpha^{-1} \subset \mathrm{Im}\, \rho_f$.

Thm 2. $\mathbb{I}$ Galois general (i.e., $\exists\, \wp \mid P_{\underline{k}}$, $\rho_\wp = \rho_{\mathbb{I}} \pmod{\wp}$ is Galois gen.) Assume $\delta \in \rho_{\mathbb{I}}(D_p)$ s.t. $\forall\ \alpha \neq \alpha'$ two roots of Sp_4, then $\alpha(\delta) \not\equiv \alpha(\delta') \pmod{m_{\mathbb{I}}}$. Assume $\overline{\rho}_{\mathbb{I}}$ also irred. Then $\exists\ \alpha \in GSp_4(\mathbb{I})$, $\exists\ l_{\mathbb{I}} \neq 0$, $l_{\mathbb{I}} \in \Lambda_2$,

$$\alpha\, \Gamma_{Sp_4}(l_{\mathbb{I}})\, \alpha^{-1} \subseteq \operatorname{Im} \rho_{\mathbb{I}}.$$

Write TY for twisted Yoshida.

$$L'_{\mathbb{I},TY} = L_{\mathbb{I},TY} \cap \Lambda_2$$

Assume $\overline{\rho}_{\mathbb{I}} = \operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_{\mathcal{M}} \pmod{\tilde{m}_{\mathbb{I}}}$

Hida family of Hilb. mod forms

$\Rightarrow$ $P \neq (p)$, $P \in (\operatorname{Spec} \Lambda_2)_{ht=1}$ then $P \supset L'_{\mathbb{I},TY}$

iff $P \supset l_{\mathbb{I}}$.