Cohomology of Bianchi Groups and Arithmetic

Set-up

$K/\mathbb{Q}$ imaginary quad. field, odd class number

$\mathcal{O}_K$ = ring of integers of K

Bianchi group $= PGL_2(\mathcal{O}_K) \overset{2}{\mid} PSL_2(\mathcal{O}_K)$ (1892 L. Bianchi)

Γ = congruence subgroup of $PGL_2(\mathcal{O}_K)$

$\Gamma \leq PGL_2(\mathcal{O}_K) \leq PGL_2(\mathbb{C})$

$\mathbb{H} = \overset{(x+iy,\, r)}{\mathbb{C} \times \mathbb{R}^{>0}}$ hyp. 3-space (symmetric space for $PGL_2(\mathbb{C})$).

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \left(\frac{(az+b)(\overline{cz+d}) + a\bar{c}\, r^2}{|cz+d|^2 + |cr|^2} , \frac{|ad-bc|\, r}{|cz+d|^2 + |cr|^2} \right)$$

Can view inside Hamiltonian quaternions:

$$(z, r) \longmapsto x + yi + rj \quad \in \left(\frac{-1, -1}{\mathbb{R}} \right)$$

$$\wr\!\downarrow$$

$$\begin{pmatrix} z & r \\ -r & \bar{z} \end{pmatrix} \in Mat_2(\mathbb{C})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z, r) = \left(\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} \begin{pmatrix} z & r \\ -r & \bar{z} \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \right) \left(\begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix} \begin{pmatrix} z & r \\ -r & \bar{z} \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & \bar{d} \end{pmatrix} \right)^{-1}$$

Γ acts on $\mathbb{H}$ via isometries; prop. discont. action.

$Y_\Gamma := \Gamma \backslash \mathbb{H}$ hyp. 3-fold, non-compact
finite volume.

- Y_Γ has no complex structure - (main obstruction)

GL_2/K K totally real = nice still have link to algebraic geometry here.

The simplest case beyond totally real is the case of imag. quad.

<u>Modern History</u>

→ Harder & his PhD students

→ Grunewald

→ Cremona

⋮

More recent years:

- Calegari
- Venkatesh
- Bergeron
- Calegari - Geraghty

<u>Cohomology</u>

$i = 1, 2$ $\quad k \geq 2$

cuspidal Bianchi mod. form

$$H^1(\Gamma, -) \simeq H^i\Big(Y_\Gamma, \underbrace{Sym^{k-2}(\mathbb{C}^2) \otimes \overline{Sym^{k-2}(\mathbb{C}^2)}}_{\text{irred reps}}\Big) \simeq S_k(\Gamma) \oplus Eis_k(\Gamma)$$

as Hecke modules

A cuspidal Bianchi modular form of wt k and level Γ is a real analytic $f : \mathbb{H} \to \mathbb{C}^{k+1}$ satisfying

- $f(\gamma z) = \mathrm{Sym}^k(j(\gamma,z)^{-1})\, f(\gamma z) \quad \forall \gamma \in \Gamma,\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

  $j(\gamma, z) = \text{"}cz+d\text{"} \in \mathrm{Mat}_2(\mathbb{C})$

- f satisfies certain diff. equation
- f vanishes at cusps.

→ f has a Fourier-Bessel expansion

→ Hecke action via Fourier expansion

→ Also have Hecke action on cohomology: using elements of commensurator of Γ $(PGL_2(K))$.

Talked about the dimension problem at Bristol.

- we do not know any formula for $\dim S_k(\Gamma)$
- recall $H^1(PSL_2(\mathbb{Z}), M) \simeq M / M^{C_2} + M^{C_3}$

  $PSL_2(\mathbb{Z}) \simeq C_2 * C_3$.

- Numerical data that suggests Bianchi modular forms are rare.

- Asymptotic result on growth of dimension

  (Lück) $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots$ s.t. $\cap \Gamma_i = \{1\}$.

$$\lim_{i\to\infty} \frac{\dim S_2(\Gamma_i)}{[\Gamma_0 : \Gamma_i]} = 0.$$

Connections with Motives:

wt 2 classical $\quad Jac(X_\Gamma) \simeq A_f \times \cdots$

One expects a similar correspondence here.

$\{$Ell curves$/K$ with no CM by $K\}/$isogeny of cond. $\mathfrak{n}$.

$\updownarrow$? L-fctns agree?

$$\left\{\begin{array}{c} f \in S_2(\Gamma_0(\mathfrak{n})) \text{ newform} \\ \text{integer Hecke eigenvalues} \end{array}\right\}$$

- Grunewald
- Cremona

gave numerical data

Can't attack this with any of classic methods. This correspondence is believed based on computational evidence.

Remark: Sometimes $\exists$ $f \in S_2(\Gamma_0(\mathfrak{n}))$ satisfying condition we want s.t. $\exists$ A/K_0 ab. surface with

$End(A) =$ quaternion algebra & $L(A/K, s) = L(f,s)^2$

These abelian surfaces are often called "fake elliptic curves".

One must ammend the above conjecture by not considering modular forms that arise as base change, and not a $\mathbb{Q}$-curve for. e. curve.

Aeni's Conjecture: Aergun

p95

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{\rho : G_K \to GL_2(\overline{\mathbb{Q}}_p)\} \arrow[d, squiggly] & \left\{\begin{matrix}(\text{"new"}) \\ \text{eigenvalue system} \\ \text{in } H^2(\Gamma, E(\mathbb{C}))\end{matrix}\right\} \arrow[l, "\text{Taylor, Harris}"', "\text{Soudry, Berger-Har}"] \\
\{\bar\rho : G_K \to GL_2(\overline{\mathbb{F}}_p)\} \arrow[r, "\text{Serre's conj.}", "?"', shift left] & \left\{\begin{matrix}\text{eigenvalue system} \\ \text{in } H^2(\Gamma, E(\overline{\mathbb{F}}_p))\end{matrix}\right\} \arrow[u, "\times"] \arrow[l, "\text{Scholze}", shift left]
\end{tikzcd}
$$

Taylor-Harris-Thorne-Lan

Scholze

Because of torsion, there can be eigenvalue systems in $H^2(\Gamma, E(\overline{\mathbb{F}}_p))$ that are not mod p reductions of eigenvalue systems of $H^2(\Gamma, E(\mathbb{C}))$.

Torsion:

$$H^1(\Gamma, E(\mathcal{O}_K)) = \text{Tor} \oplus \text{Free}$$

There is a lot of torsion here!

Bergeron-Venkatesh give asymptotic results about torsion

Calegari-Venkatesh give Jacquet-Langlands mod p.

Γ Bianchi

$\Big\updownarrow$ J-L

Γ' cocompact arithmetic $\leq PGL_2(\mathbb{C})$

$$\left\{\begin{matrix} H^1(\Gamma, \overline{\mathbb{F}}_p) \\ \text{e.v systems}\end{matrix}\right\} \rightsquigarrow \left\{\begin{matrix} H^1(\Gamma', \overline{\mathbb{F}}_p) \\ \text{e.v. systems}\end{matrix}\right\}$$

related..

Possibly the torsion has nice functoriality, etc.