

Set-up

K/\mathbb{Q} imaginary quad. field, odd class number

$\mathcal{O}_K =$ ring of integers of K

Bianchi group = $PGL_2(\mathcal{O}_K)$ (1892 L. Bianchi)
 \downarrow
 $PSL_2(\mathcal{O}_K)$

$\Gamma =$ congruence subgroup of $PGL_2(\mathcal{O}_K)$

$\Gamma \leq PGL_2(\mathcal{O}_K) \leq PGL_2(\mathbb{C})$

$H = \mathbb{C} \times \mathbb{R}^{>0}$ hyp. 3-space (symmetric space for $PGL_2(\mathbb{C})$)
($x+iy, r$)

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) = \left(\frac{(az+b)(\overline{cz+d}) + a\overline{c}r^2}{|cz+d|^2 + |cr|^2}, \frac{|ad-bc|r}{|cz+d|^2 + |cr|^2} \right)$$

Can view inside Hamiltonian quaternions:

$$(z, r) \longmapsto x+yi+rj \in \left(\frac{-b-1}{\mathbb{R}} \right)$$



$$\begin{pmatrix} z & r \\ -r & \overline{z} \end{pmatrix} \in Mat_2(\mathbb{C})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z, r) = \left(\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} z & r \\ -r & \overline{z} \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & \overline{b} \end{pmatrix} \right) \left(\begin{pmatrix} c & 0 \\ 0 & \overline{c} \end{pmatrix} \begin{pmatrix} z & r \\ -r & \overline{z} \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & \overline{d} \end{pmatrix} \right)^{-1}$$

Γ acts on H via isometries; prop. discont. action.

$Y_\Gamma \cong \Gamma \backslash \mathbb{H}$ hyp. 3-fold, non-compact
finite volume.

Bergman
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Y_Γ has no complex structure - (main obstruction)

GL_2/K K totally real = nice still have link to
algebraic geometry here.

The simplest case beyond totally real is the case of imag. quad.

Modern History

- Harder & his PhD students
- Gunnarsson
- Cremona
- ⋮

More recent years:

- Calegari
- Venkatesh
- Bergeron
- Calegari - Geraghty

Cohomology

$i=1, 2$ $K \geq 2$

$$H^i(\Gamma, -) \cong H^i(Y_\Gamma, \underbrace{\text{Sym}^{K-2}(\mathbb{C}^2) \otimes \overline{\text{Sym}^{K-2}(\mathbb{C}^2)}}_{\text{irred reps}}) \cong S_K(\Gamma) \oplus \mathcal{E}_{15_K}(\Gamma)$$

↗ cuspidal
Bianchi
mod. form

 ↖ as Hecke modules

A cuspidal Bianchi modular form of wt k and level Γ

Aengun

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is a real analytic $f: \mathbb{H} \rightarrow \mathbb{C}^{k+1}$ satisfying

$$\bullet f(\gamma z) = \text{Sym}^k(j(\gamma, z)) f(z) \quad \forall \gamma \in \Gamma, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$j(\gamma, z) = "cz+d" \in \text{Mat}_2(\mathbb{C})$$

$\bullet f$ satisfies certain diff. equation

$\bullet f$ vanishes at cusps.

$\rightarrow f$ has a Fourier-Bessel expansion

\rightarrow Hecke action via Fourier expansion

\rightarrow Also have Hecke action on cohomology: using elements of commensurator of Γ ($\text{PGL}_2(K)$).

Talked about the dimension problem at Bristol.

\bullet we do not know any formula for $\dim S_k(\Gamma)$

$$\bullet \text{ recall } H^1(\text{PSL}_2(\mathbb{Z}), M) \cong \frac{M}{M^{c_2} + M^{c_3}}$$

$$\text{PSL}_2(\mathbb{Z}) \cong C_2 * C_3.$$

\bullet Numerical data that suggests Bianchi \mathbb{H} modular forms are rare.

\bullet Asymptotic results on growth of dimension

$$(\text{Lück}) \quad \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \dots \text{ s.t. } \cap \Gamma_i = \{2, 3\}.$$

$$\lim_{i \rightarrow \infty} \frac{\dim S_2(\Gamma_i)}{[\Gamma_0 : \Gamma_i]} = 0.$$

Benjamin
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Connections with Motives:

wt 2 classical $\text{Jac}(X_f) \cong A_f \times \dots$

One expects a similar correspondence here.

{ Ell curves / K with no CM by K } / isogeny
of cond. $\#1$.

\updownarrow ? L -fctns agree?

{ $f \in S_2(\Gamma_0(N))$ newform }
integer Hecke eigenvalues

- Grosswald gave numerical data
- Cremona

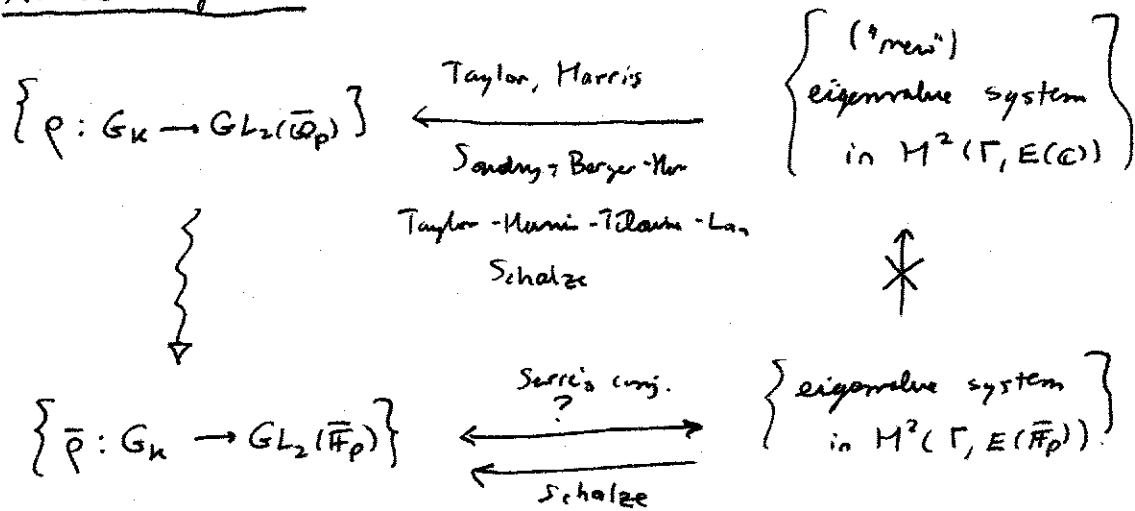
Can't attack this with any of classic methods. This correspondence is believed based on computational evidence.

Remark: Sometimes $\exists f \in S_2(\Gamma_0(N))$ satisfying conditions we want s.t. $\exists A/K_0$ ab. surface with $\text{End}(A) = \text{quaternion algebra}$ & $L(A/K, s) = L(f, s)^2$. These abelian surfaces are often called "fake elliptic curves".

One must amend the above conjecture by not considering modular forms that arise as base change, and not a \mathbb{Q} -curve for, e.g. curve.

Serre's Conjecture:

Arun
p95



Because of torsion, there can be eigenvalue systems in $H^2(\Gamma, E(\mathbb{F}_p))$ that are not mod p reductions of eigenvalue systems of $H^2(\Gamma, E(\mathbb{O}))$.

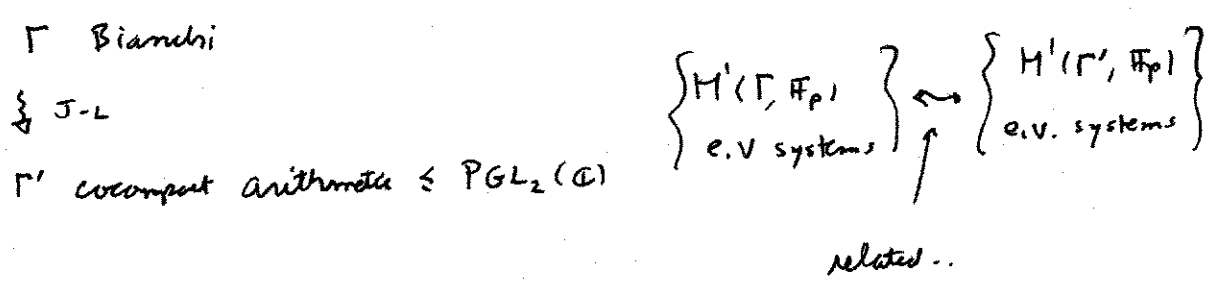
Torsion:

$$H^1(\Gamma, E(\mathbb{O}_K)) = \text{Tor} \oplus \text{Free}$$

There is a lot of torsion here!

Bergeron-Venkatesh give asymptotic results about torsion

Calegari-Venkatesh give Jacquet-Langlands mod p .



Possible the torsion has nice functoriality, etc.