Non-vanishing of fundamental Fourier coefficients

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Siegel cusp forms $f : \{Z \in Mat_2(\mathbb{C}) : Z = {}^tZ, \, Im(Z) > 0\} \to \mathbb{C}$ holomorphic s.t.

- $f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \leq Sp_4(\mathbb{Q})$
- f vanish at cusps.

$$f(Z) = \sum_{\substack{T = {}^tT, \, T > 0 \\ \text{half-int.}}} a(T)\, e(tr(TZ))$$

Hecke operators

$p \nmid N \quad T_p := \Gamma \begin{pmatrix} I & \\ & pI \end{pmatrix} \Gamma \quad , \quad T_{p^2} = \Gamma \begin{pmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{pmatrix} \Gamma$

$p \mid N \quad U_p = \Gamma \begin{pmatrix} I & \\ & pI \end{pmatrix} \Gamma.$

We will be interested in

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : C \equiv 0 \pmod N \right\}$$

$$\Gamma = \Gamma^{para}(N) = \left\{ \begin{pmatrix} * & N & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} : * \in \mathbb{Z} \right\}$$

Paramodular conjecture

$$\left\{ \begin{array}{l} \text{isogeny classes of abelian} \\ \text{surfaces } A/\mathbb{Q} \text{ with} \\ \text{conductor } N, \, End_{\mathbb{Q}} = \mathbb{Z} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} f \in S_2(\Gamma^{para}(N)) \text{ newforms} \\ \text{w/ rational evs that are not} \\ \text{Gritsenko lifts / up to scalar mult.} \end{array} \right\}$$

$$L(A,s) = L(f,s).$$

Böcherer (conjecture): $0 \neq F \in S_k(Sp_4(\mathbb{Z}))$, π_F associated auto rep., then $\exists\ C_F$ (depends only on F) $\forall\ K = \mathbb{Q}(\sqrt{D})$

$D<0$ fund disc

$$\Bigg(\sum_{T \in \{T \text{ s.t. } \det T = D\}/SL_2(\mathbb{Z})} \frac{a(T)}{\varepsilon(T)} \Bigg)^2 = C_F |D|^{k-1}\, L(1/2, \pi_F \times \chi_D)$$

$$\varepsilon(T) = \big| \{ A \in SL_2(\mathbb{Z}) : A T\, {}^tA = T \} \big|$$

$$LHS = \Big(\sum_{c \in Cl_K} a(c) \Big)^2 \frac{1}{\omega(K)^2}$$

- An average of fundamental Fourier coeffs by certain character Λ corresponds to certain Rankin-Selberg L-functions. So if there exists fundamental Fourier coeff $a(S) \neq 0$, then certain central value of L-function is non-zero as well. (under conjecture)

What happens if $F \in S_k(\Gamma_0(N))$ or $F \in S_k(\Gamma^{para}(N))$?

Thm (Saha-Schmidt 2012): Let $0 \neq F \in S_k(\Gamma_0(N))$, k even, $k>2$, N sq. free, an eigenfctn of U_p for all $p \mid N$. Then for any $0 < \delta < 5/8$ one has

$$\big| \{ 0 < d < X : d \text{ odd, sq. free, } \exists\ S,\ \operatorname{disc} S = -d,\ a(F,S) \neq 0 \} \big| \gg_{F,\delta} X^{\delta}.$$

In particular, F has infinitely many non-zero fund. Fourier coeffs.

Important ingredient:

1) Yamana (2009): $0 \neq F \in S_k(\Gamma_0(N)) \Rightarrow \exists\, a(F,T) \neq 0$ w/ $\mathrm{cont}(T) \mid N$, $\mathrm{cont}\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} = \gcd(a,b,c)$.

2) Saha (2012): $0 \neq f \in S_{k+1/2}(\Gamma_0(4N))$, $k \geq 2$, N sq. free
$$\Rightarrow \exists\, \delta > 0 \text{ s.t. } |\{0 < d < X : d \ \square\text{-free}, a(f,d) \neq 0\}| \gg_{f,d} X^{\delta}.$$

In the case $\Gamma = \Gamma^{para}(N)$:

Thm (M.): Let $0 \neq F \in S_k(\Gamma^{para}(N))$, $N \geq 2$ $\square$-free, Hecke e.f. of T_p, T_{p^2}, U_p, μ_N, Then

(μ_N = $\frac{1}{\sqrt{N}}\begin{pmatrix} & & 1 & \\ & & & 1 \\ N & & & \\ & N & & \end{pmatrix}$)

1) $\exists\ T$ w/ $\mathrm{cont}(T) = 1$ s.t. $a(F,T) \neq 0$.

2) if $k > 2$, F has infinitely many non-zero fund. Fourier coeffs

~~Proof sketch~~:

Cor: With assumptions as above, π_F assoc. auto rep., Then π_F has a fund. Bessel model.

Proof (sketch):

1) If $p \| N$, $F|_k U_p = \lambda F$, then

$$\lambda a(T) = p^{-k+3} a(pT) + p^k a(\tfrac{1}{p}T) - a\left(\tfrac{1}{p}\begin{pmatrix} \alpha p & 1 \\ -Np & p \end{pmatrix} T \begin{pmatrix} \alpha p & -Np \\ 1 & p \end{pmatrix}\right)$$
$$+ \cdots$$

Important: All matrices, except a pT, have disc. that divides disc T. So if $a(pT) \neq 0$, then $\exists\, T'$ s.t. disc $T' \mid$ disc T s.t. $a(T') \neq 0$.

$\rightsquigarrow \exists\, S$ s.t. $\gcd(\text{cont}(S), N) = 1$ and $a(S) \neq 0$.

2) Evdokimov (1976) if $F|_k(T_p + T_{p^2}) = \lambda F$, then

$$\lambda a(T) = a(pT) + p^{2k-3} a(\tfrac{1}{p}T) + p^{k-2} \sum_{U \in R_p < \Gamma_0(N)} a\left(\tfrac{1}{p}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} U T\, {}^t U \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right).$$

Again, for $p \nmid N$, we can get $\exists\, S$ w/ $\text{cont}(S) = 1$ s.t. $a(S) \neq 0$.

3) Use 2) to obtain there exists an odd prime $p \nmid N$ s.t. Jacobi-Fourier coeff. of F, ϕ_{Np} $\neq 0$

$$F(Z) = \sum_{\substack{m>0,\ N|m \\ 4nm - r^2 > 0}} c(n,r,m)\, e(n\tau) e(rz) e(m\tau')$$

$Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ $\qquad$ $c(n,r,m) = a\left(\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}\right)$

$$= \sum_{\substack{m>0 \\ N|m}} \phi_m(\tau, z)\, e(m\tau').$$

Define

$$h(z) = \sum_{D=1}^{\infty} \underbrace{\sum_{\substack{0 \le r^2 < 2Np \\ r^2 \equiv -D \pmod{4Np}}} c\left(\frac{D+r^2}{4Np}, r, Np\right)}_{c(D)} e(Dz)$$

$\phi_{Np} \neq 0 \Rightarrow 0 \neq h \in M_{k-1/2}(4Np)$.

Saha $\Rightarrow$ There are ∞'ly many D ~~sth~~ sq. free, ~~odd~~ s.t. $c(D) \neq 0$.