Congruences and R=T theorems (joint w/ Berger and Kramer)

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Thm (Ramakrishna, Skinner-Wiles): Let $p \geq 3$ be a prime, $[\mathbb{F} : \mathbb{F}_p] < \infty$.

Let $\chi : G_{\mathbb{Q}} \to \mathbb{F}^\times$ be a cont. char. s.t. $\chi^2 \neq 1$, $\chi \neq \mathrm{cycl}^{\pm}$ and s.t. the $\mathbb{F}_p$-span of the image of χ is all of $\mathbb{F}$.

Let $\rho_0 : G_{\mathbb{Q}} \to GL_2(\mathbb{F})$ be a cont. rep. of the form

$\rho_0 = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix} \not\cong \chi \oplus 1$, ρ_0 odd, $\rho_0|_{D_p} \neq 1$. Then ρ_0 is modular.

Our set-up:

F = im. quad. ext of $\mathbb{Q}$, $p > 3$, $p \nmid \# Cl_F$ $\mathrm{disc}_F \neq 3, 4$.

$\chi : G_F \to F^\times$ a cont. char. s.t. $\chi(cgc) = \chi(g)^{-1}$ $\forall g \in G_F$, c = complex conj

(i.e., the char is anti-cyclotomic)

Σ = a finite set of places of F containing $\{\mathfrak{p} | p\}$, places where χ is ramified. Set

$$\rho_0 : G_\Sigma \to GL_2(\mathbb{F})$$

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix} \not\cong 1 \oplus \chi.$$

(G_Σ = absolute Galois group of ext max ext unram. outside Σ)

Questions:

Q1: Does ρ_0 arise as a reduction of $\rho_\pi : G_F \to GL_2(\mathcal{O})$ for some auto rep. π of $GL_2(\mathbb{A}_F)$?

$[E/\mathbb{Q}_p$ = finite ext. $\mathcal{O} \subset E$, $\varpi \in \mathcal{O}$, $\mathcal{O}/\varpi = \mathbb{F}]$.

Q2: Do all lifts $\rho : G_\Sigma \to GL_2(\mathcal{O})$ of ρ_0 satisfying some local conditions arise from some π?

Q1 appears to be hard.

~~Problem~~

Note that each ρ_0 can be regarded as an element of $H^1(F, \chi^{-1})$. If we also impose that ρ_0 should satisfy certain local condition, then ρ_0 can be regarded as an element of the Selmer group $H^1_\Sigma(F, \chi^{-1})$.

Thm1 (Berger-K.): There exists an $\mathbb{F}$-basis of $H^1_\Sigma(F, \chi^{-1})$ which is modular.

Can replace Q1 with:

Q1': Are all elements of $H^1_\Sigma(F, \chi^{-1})$ modular?

If one fixes the level of the auto forms, then in general no. If one is allowed to vary the level, we don't know.

What about Q2?

Thm2: (Berger-K.) If $\dim_{\mathbb{F}} H^1_\Sigma(F, \chi^{-1}) = 1$, then every (minimal) lift $\rho : G_\Sigma \to GL_2(\mathcal{O})$ (crys. or ordinary at p) of ρ_0 is modular.

Proof: Recall $\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix}$.

$$\begin{array}{ccc} R_{\rho_0} & \twoheadrightarrow & \mathbb{T} \\ \downarrow & & \downarrow \\ R_{\rho_0}/I_{\rho_0} & \twoheadrightarrow & \mathbb{T}/J \end{array}$$

M.C. due to Rubin gives $\# \left(R_{\rho_0}/I_{\rho_0}\right) \leq \# \mathcal{O}/L\text{-value}$.

Eisenstein congruence (Berger) gives $\# \mathbb{T}/J \geq \mathcal{O}/L\text{-value}$.

So the bottom arrow is an isom. Use comm. alg. to get top

arrow is also an isom. ∎

What if $\dim_{\mathbb{F}} H^1_\Sigma(F, \chi^{-1}) > 1$?

Difficulties:

(1) R_{ρ_0}, $R^{red}_{\rho_0}$, $R^{red}_{\rho_0}/I^{red}_{\rho_0}$ are not finitely generated

over $\mathbb{Z}_p$.

(2) There is no map $R_{\rho_0} \to \mathbb{T}$.

To remedy (1) we consider

$$R^0_{\rho_0} = \text{image of } R^{red}_{\rho_0} \text{ in } R^{red}_{\rho_0} \otimes \overline{\mathbb{Q}}_p.$$

To remedy (2) we define a quotient

$\mathbb{T}_{\rho_0}$ of $\mathbb{T}$ corresponding to those π for which

$\rho_\pi \bmod \varpi \simeq \rho_0$.

Even with this, we still only get a map

$$\cancel{R}\ R_{\rho_0}^{\circ,\mathrm{tr}} \twoheadrightarrow \mathbb{T}_{\rho_0}.$$

We'd hope to recover "$R/I \simeq \mathbb{T}/J$".

Let $\mathcal{B}$ be a modular basis of $H^1_\Sigma(F,\chi^{-1})$, say $\mathcal{B}=\{\rho_0,\dots,\rho_n\}$.

How to piece together the various R_{ρ_i}'s and $\mathbb{T}_{\rho_i}$'s?

Consider

$$\underbrace{\prod_{i=0}^{n} R_{\rho_i}^{\circ,\mathrm{tr}}/I_{\rho_i}}_{\substack{\#\ \leq\ \mathcal{O}/L\text{-value} \\ \text{by M.C. due to} \\ \text{Rubin.}}} \twoheadrightarrow \prod_{i=0}^{n} \mathbb{T}_{\rho_i}/J_i \overset{?}{\longleftrightarrow} \underset{\#\geq\ \mathcal{O}/L\text{-value.}}{\mathbb{T}/J}$$

$$\begin{array}{ccc}
\mathbb{T} & \longrightarrow & A=\prod_{i=0}^{n} A_i \\
\downarrow & & \downarrow \varphi_i \quad \text{proj.} \\
\mathbb{T}_{\rho_i} & \longrightarrow & A_i=\prod_{\substack{\mathbb{T}\twoheadrightarrow\mathcal{O} \\ \text{corresp to} \\ \rho_i}} \mathcal{O}
\end{array}$$

<u>Thm 3 :</u> (Berger - K. - Kramer): Let $A_i=\prod_{j=1}^{n_i}\mathcal{O}$, $A=\prod_{i=0}^{n} A_i$,

$\varphi_i : A \twoheadrightarrow A_i$. Let $T \subseteq A$ be a local complete $\mathcal{O}$-subalg.

of full rank and $J \subseteq T$ an ideal of finite index.

If $\forall i$, $\varphi_i(J)$ is a principal ideal of $\varphi_i(T)$,

then
$$\# \prod_{i=0}^{n} \frac{\varphi_i(T)}{\varphi_i(J)} \geq \# T/J .$$

Furthermore, equality holds iff J is principal.

Thm 4: (Berger-K.) Let $\rho_0 = \begin{bmatrix} 1 & * \\ 0 & \chi \end{bmatrix} : G_\Sigma \to GL_2(\mathbb{F})$ be modular. Let $\mathcal{B}$ be a modular basis of $H^1_\Sigma(F, \chi^{-1})$ s.t. $\rho_0 \in \mathcal{B}$. Assume $\forall \rho_i \in \mathcal{B}$, $R^{0,\mathrm{tr}}_{\rho_i}$ is a f.g. $\mathbb{Z}_p$-module. If $\mathcal{B}$ is unique up to scaling or p annihilate $H^1_\Sigma(F, \tilde{\chi} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, then any lift $\rho : G_\Sigma \to GL_2(\mathcal{O})$ satisfying some local cond. (crys or ord at p) of ρ_0 is modular.

Cor: J is principal.

Applications of Thm 3:

Prop. 1: (Berger-K.-Kramer): Let $E_2^*(N) = 1 - N + 24 \sum_{n=1}^{\infty} \Big(\sum_{\substack{0<d\mid n \\ (d,N)=1}} d \Big) q^n \in M_2(N)$, N prime, coprime to p, and let $\{f_1, \dots, f_r\}$ be a basis of eigenforms for $S_2(N)$. Let $(\mathcal{O} \subseteq) E$ be the

finite ext. of $\mathbb{Q}_p$ generated by the Hecke eigenvalues of all the f_i's. For $i \in \{1, \dots, r\}$ let m_i be the largest pos. integer m s.t. $f_i \equiv E_2^* \pmod{\varpi^m}$, then

↑
e.v. congruence
away from N

$$\frac{1}{e_{\mathcal{O}/\mathbb{Z}_p}} \sum_{i=1}^{r} m_i = \mathrm{val}_p\left(\mathrm{Num}\left(\frac{N-1}{12}\right)\right).$$

(Mazur showed if RHS > 0, then LHS > 0.)

<u>Prop. 2:</u> $k \in 2\mathbb{Z}_+$, $p > k$, $f \in S_{2k-2}(SL_2(\mathbb{Z}))$. Let F_f be the SK lift of f. Combining theorem Thm 3 with the work of J. Brown we get

$$\frac{1}{e_{\mathcal{O}/\mathbb{Z}_p}} \sum_{F \in S_k^{\mathrm{Sies}}(Sp_4(\mathbb{Z}))} m_i \geq \#\left(\mathcal{O}/L^{\mathrm{alg}}(k,f)\right).$$

↑
lin indep e.f.
⊥ to SK-lifts.

$F \equiv F_f \pmod{\varpi^{m_F}}$ ← max. such.