Congruence primes via higher L-functions

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(based on discussion w/ H. Katsurada)

Setting:

$\Gamma = \Gamma_0(N) \subset \Gamma_n = Sp(n,\mathbb{Z}) \circlearrowright \mathbb{H}_n$.

$S_\ell^n(\Gamma)$ = space of Siegel cusp forms for Γ, weight ℓ.

$S_\ell^n(\Gamma) = S_\ell^n(\Gamma)(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

f eigenform $\rightsquigarrow$ standard L-fctn.

$$L^{(N)}(f, st, s) = \prod_{p \nmid N} \frac{1}{1-p^{-s}} \left(\prod_{i=1}^{n} \frac{1}{(1-\alpha_{i,p} p^{-s})(1-\alpha_{i,p}^{-1} p^{-s})} \right)$$

Critical values for $L(f, st, s, \chi)$; $\ell = k+v$, $v \geq 0$

are $\{k-n > 0 \mid \ell = k+v,\ k \text{ even}\} \cup \{s \leftrightarrow 1-s\}$ χ even

$\{k-n > 0 \mid \ell = k+v,\ k \text{ odd}\} \cup \{s \leftrightarrow 1-s\}$ χ odd.

χ arb. Dirichlet character.

For all critical values r

$$\frac{L^{(N)}(f, st, r, \chi)}{\pi^{?} \langle f, f \rangle} \in \bar{\mathbb{Q}}$$

if $f \in S_\ell^n(\Gamma)(\bar{\mathbb{Q}})$.

Def: $f \in S_\ell^n(\Gamma)$ eigenform (away from level)

$\mathcal{F}$ = set of e.f. in $S_\ell^n(\Gamma)$, including f, all have same Hecke eigenvalues.

K = field containing all e.v.'s of f

$\wp$ a prime of K. We say $\wp$ is a <u>congruence prime for f wrt $\mathcal{F}$</u> iff $\exists\ g \in \mathcal{F}^{\perp}$ Hecke eigenform s.t.

$$\lambda_f(T) \equiv \lambda_g(T) \pmod{\tilde{\wp}}$$

for all good Hecke operators, $\tilde{\wp} \mid \wp$.

Criteria for congruence primes

~~[illegible]~~

Need to normalize the f. Write $f = \sum_{S \in P_n(\mathbb{Z})} a_f(S)\, e^{2\pi i \operatorname{tr}(SZ)}$.

$a_f(S) \in K$. $c\ell(f) = \mathcal{O}_K$ - module generated by all the $a_f(S)$.

<u>Thm (Katsurada, et. al):</u> f as above, r critical, $\wp$ a prime of K (away from denominators of Fourier coeffs of "the" Eisenstein series) if $v(\omega, r)$

$$v_\wp\left(\frac{L(f, st, r)}{\pi^{*} < f, f >}\, c\ell(f)^2\right) < 0$$

then $\wp$ is a congruence prime for f wrt $\mathcal{F} = \{f\}$.

To avoid trivial cases,

$\mathfrak{a}_{\mathcal{F}} = \mathcal{O}_K$ - module generated by

$$\left\{ \sum_{i=1}^{t} \frac{a_{f_i}(S)\, a_{f_i}(T)}{\langle f_i, f_i \rangle} \;\middle|\; S, T \in P_n(\mathbb{Z}) \right\}$$

$\mathcal{F} = \{ f = f_1, \dots, f_t \}$

if $\quad \nu_{\wp}\left(\gamma(\infty, r)\, \dfrac{L(f, st, r)}{\pi^{*}}\, \mathfrak{a}_{\mathcal{F}} \right) < 0 \quad \Rightarrow \quad \wp$ is a

congruence prime for f wrt $\mathcal{F}$.

<u>Variant:</u> f as before, level 1, χ char. mod N.

if $\quad \nu_{\wp}\left(\gamma(\infty, r)\, \dfrac{L(f, st, r, \chi)}{\pi^{*} \cdot \langle f, f \rangle} \right) < 0 \quad \Rightarrow \quad \wp$ is

a congruence prime for f wrt $\mathcal{F} = \{f\}$.

<u>Ingredients of proof:</u>

- integral representation of standard L-function (doubling method)
- holomorphic diff. operators (to get all critical values)
  (at $s = 0$, $s = \frac{2n-1}{2} - k$ get holo Siegel E.S.)
- exterior twist (wrt variables to be put equal to zero afterwards)
- ~~[illegible] lemma on~~ integrality of Fourier expansion of E.S.
- elementary lemma of Kurokawa - Katsurada.

<u>Triple product:</u>

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$f, g, h \in S_k^1(SL_2(\mathbb{Z}))$

r critical

$$v_{\wp}\left(v_\infty \frac{L(f\otimes g\otimes h, r)}{\pi^{*} \langle f,f\rangle\langle g,g\rangle\langle h,h\rangle}\right) < 0$$

$\Rightarrow$ $\wp$ is a congruence prime for f, g, or h.