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3-28-15

The mean number of 3-torsion elements in the class groups and ideal groups of quadratic orders

K = # field, $\mathcal{O}_K$ = ring of integers

$\mathcal{O} \subseteq \mathcal{O}_K$ subring s.t. $\mathrm{Frac}(\mathcal{O}) = K$.

order

$\mathcal{I}(\mathcal{O})$ = inv. fractional ideals of $\mathcal{O}$

$P(\mathcal{O})$ = principal ideals

$Cl(\mathcal{O}) = \mathcal{I}(\mathcal{O})/P(\mathcal{O})$ class group

if $\mathcal{O} = \mathcal{O}_K$: $\mathcal{I}(\mathcal{O}_K)$ cannot have torsion (it is a Dedekind domain)

In general for orders, can have torsion in $\mathcal{I}(\mathcal{O})$.

<u>Example:</u> $\mathcal{O} = \mathbb{Z}[\sqrt{-11}]$ $\quad I = (2, \frac{1-\sqrt{-11}}{2})$. Then

$I^3 = \mathcal{O}$.

<u>Question:</u> How does this torsion showing up in $\mathcal{I}(\mathcal{O})$ affect distribution of torsion on average when you move from maximal orders to all orders?

<u>Thm (Davenport-Heilbronn):</u> The mean number of 3-torsion elements in class groups of quadratic fields ordered by discriminant is

$$\begin{cases} 4/3 & \text{imag. quad.} \\ 2 & \text{real quad.} \end{cases}$$

<u>Thm (Bhargava-V.):</u>

$$\lim_{X \to \infty} \frac{\sum_{\substack{\mathcal{O} \text{ s.t.} \\ \mathrm{disc}(\mathcal{O}) < X}} |\mathcal{I}_3(\mathcal{O})|}{\sum_{\mathrm{disc}(\mathcal{O}) < X} 1} = \frac{\zeta(2)}{\zeta(3)} \sim 1.368 > 1$$

Cor: The mean size of 3-torsion elements in class groups of quad. orders is

$$\begin{cases} 1+ \zeta(2)/\zeta(3) & \text{real} \\ 1+\frac{1}{3}\frac{\zeta(2)}{\zeta(3)} & \text{imag.} \end{cases}$$

Pf:

Ⓐ need a direct parameterization of 3-torsion ideal classes of quadratic orders

Prop.: $\left\{(\mathcal{O}, I, \delta) \ \begin{matrix} I \text{ inv. frac. ideal} \\ \delta \in \mathrm{Frac}\,\mathcal{O} \end{matrix} \ \text{s.t.} \ I^3 = (\delta)\right\}$

$\downarrow$

$\xleftrightarrow{1-1} \left\{(\mathcal{O}, I, \delta)'s \right\}/\sim$

$\xleftrightarrow{1-1} \left\{ f(x,y) = ax^3+3bx^2y+3cxy^2+dy^3 : a,b,c,d \in \mathbb{Z} \right\}/GL_2(\mathbb{Z})$

integer matrix or (1-3-3-1 forms)

$\gamma f(x,y) = f((x,y)\gamma)$, $\gamma \in GL_2(\mathbb{Z})$.

where $(\mathcal{O}, I, \delta) \sim (\mathcal{O}', I', \delta')$ iff $\exists\, \phi : \mathcal{O} \xrightarrow{\sim} \mathcal{O}'$ and $\kappa \in \mathrm{Frac}(\mathcal{O})$

s.t. $I' = \kappa\,\phi(I)$, $\delta' = \kappa^3 \phi(\delta)$.

Rmk: ~~Given~~ If $f \longleftrightarrow (\mathcal{O}, I, \delta)$, then $\mathrm{disc}(f) = -27\,\mathrm{Disc}(\mathcal{O})$.

We define $(\mathcal{O}, I, \delta)\cdot(\mathcal{O}', I', \delta') = (\mathcal{O}, II', \delta\delta')$

since we restrict to I invertible.

Def: $H(\mathcal{O}) = \{(\mathcal{O}, I, \delta)/\sim \text{ valid triples}\}$

We have
$$1 \longrightarrow \mathcal{O}^\times_{Nm=1}/(\mathcal{O}^\times_{Nm=1})^3 \longrightarrow H(\mathcal{O}) \longrightarrow Cl_3(\mathcal{O}) \longrightarrow 1.$$
$$(\mathcal{O}, I, \delta) \longmapsto [I]$$

Rmk: $\mathrm{Disc}(\mathcal{O}) < -3$, $H(\mathcal{O}) = Cl_3(\mathcal{O})$.

We can define $\varphi: Cl_3(\mathcal{O}) \longrightarrow H(\mathcal{O})$
$$I \longmapsto (\mathcal{O}, I, 1).$$

Q: What is the image of φ?

φ induces an isom. $Cl_3(\mathcal{O}) \longrightarrow H_{red}(\mathcal{O}) = \left\{\begin{matrix}(\mathcal{O}, I, \delta) \text{ that correspond} \\ \text{to reducible } f\end{matrix}\right\}$

(B) Use this parameterization to count 3-torsion ideal classes while varying $\mathcal{O}$.

real binary cubic forms: $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$
$a, b, c, d \in \mathbb{R}$

Davenport: $\exists$ a noncompact fundamental domain for the action of $GL_2(\mathbb{Z})$

Projecting to $\mathbb{R}^2$ so we can draw it:

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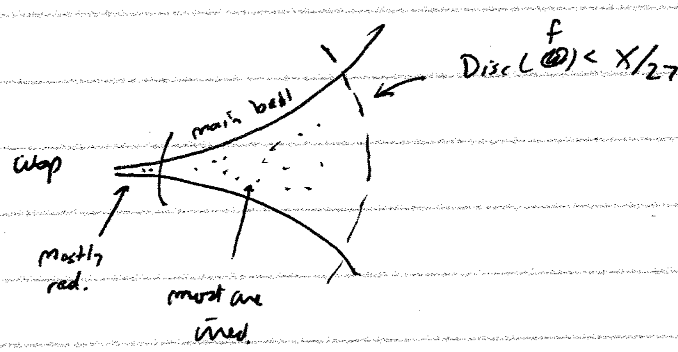

volume of main ball $\sim \sum_{-X < \text{Disc}(\mathcal{O}) < 0} |Cl_3(\mathcal{O})| - |Cl_3(\mathcal{O})|$

Computing the volume:
$$\lim_{X \to \infty} \frac{\sum_{-X<D<0} |Cl_3(\mathcal{O})| - |Cl_3(\mathcal{O})|}{\sum_{-X<D<0} 1} = 1.$$

Now restrict to maximal orders, we get the same thing.

Looking at maximal orders $Cl_3(\mathcal{O}_K) = 1$. This recovers the theorem of Davenport-Heilbronn.

volume of cusp $\sim \sum_{-X<\text{Disc}\,\mathcal{O}<0} |Cl_3(\mathcal{O})|$

$= \zeta(2)/\zeta(3)$.

<u>Generalizations:</u> 2 cubic orders

The mean size of $|Cl_2(\mathcal{O})| - \frac{1}{4}|I_2(\mathcal{O})|$ when orders are ordered by discriminant is 1.

(independent of the acc family that you're averaging over)

Obs: Fix n odd.

Melanie Wood defined a family of n-ic orders that are parameterized by binary n-ic forms.

<u>Ongoing work:</u> (Ho, Shankar, V.)

$$|Cl_2(\mathcal{O})| - \frac{1}{2^?}|Cl_2(\mathcal{O})| \overset{\text{is}}{=} 1.$$ on average for Wood's family (ordered by disc.)