Pitale
3-29-15
pg 1

Special Values of L-functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

$$\zeta(2) = \pi^2/6$$

$$\zeta(4) = \pi^4/90$$

$$\vdots$$

If $k \in \mathbb{Z}_{>0}$, $\frac{\zeta(2k)}{\pi^{2k}} \in \mathbb{Q}$.

Thm (Shimura 1976): $f \in S_k(N, \chi)$, primitive (Hecke eigenform, newform, $a(1) = 1$). There exist $C^{\pm}(f) \in \mathbb{C}$ s.t. for all $1 \le m \le k-1$

$$\frac{L(m,f)}{(2\pi i)^m C^{\pm}(f)} \in \overline{\mathbb{Q}}. \qquad \left(\begin{array}{ll} + & m \text{ even} \\ - & m \text{ odd} \end{array}\right)$$

($C^+(f)C^-(f) \leadsto \langle f, f\rangle$.) In fact, this lands in $\mathbb{Q}(\{a_n\}) = K_f$, a number field.

Deligne's conjecture (1979): Let M be a motive, $L(s,M)$ the motivic L-function, m a "critical point" for $L(s,M)$, then $\exists$ $C^{\pm}(M)$ s.t.

$$\frac{L(m,M)}{C^{\pm}(M)} \in \overline{\mathbb{Q}}.$$

$C^{\pm}(M)$ are called Deligne's periods.

Shimura's result is a special case of Deligne's conjecture.

Pitale
3-29-15
pg 2

The special value arise at integers where neither gamma factor in the functional equation has a pole.

Thm (Shimura 1976): Let $f \in S_k(N,\chi)$, $g \in S_\ell(N,\psi)$ both primitive, let $\ell < k$. Then for all $\ell \le m \le k-1$

$$\frac{L(m, f \times g)}{(2\pi i)^{2m+k-1}\langle f, f\rangle} \in K_f K_g.$$

Main idea of proof: Write integral representation:

$$(4\pi)^{-s}\Gamma(s) L(s, f\times g) = \int_{\Gamma_0(N)\backslash \mathfrak{H}} \overline{f}(z)\, g(z) \overbrace{E_{\ell-k,N}(z, s+1-k)}^{\text{Eisenstein series}} y^{s+1} \frac{dx\,dy}{y^2}$$

$$L(k-1-r, f\times g) \rightsquigarrow \langle \overline{f}, g\, \delta^{(r)} \overset{\text{nice and arithmetic}}{\widetilde{E(z,0)}} \rangle$$

differential operator

$\delta^{(r)}E(z,0)$ nearly holomorphic mod. forms.
(polynomial in y at worst)

Deligne's conjecture for $L(s,f,\mathrm{Sym}^\ell)$. This is a deg $\ell+1$ L-function. For p not dividing the level it is easy to write down Euler factors. At bad primes it is not so obvious.

For $\ell = 3$ we have the conjecture says:

$k \leq m \leq 2(k-1)$

$$\frac{L(m, f, \mathrm{Sym}^3)}{(2\pi i)^{2m} \, C^{\pm}(f)^3 \, C^{\mp}(f)} \in K_f .$$

For other l one can see the survey paper of Raghuram - Shahidi.

__Known:__

$l=2$: Sym^2 Sturm (1980)

$l=3$: Sym^3 Garrett - Harris (1993) Special case of triple product L-function result.

Sym^5, Sym^7 Raghuram (2008) right-most critical point.

Sym^4 work in progress P. - Saha - Schmidt

Ramakrishnan - Shahidi (2007) : $f \rightsquigarrow F$ holo. vector-valued Siegel cusp form of degree 2

$$L(s, f, \mathrm{Sym}^3 f) = L(s, F, \mathrm{spin})$$

(not endoscopic!)

Check: $L(s, f, \mathrm{Sym}^4) = L(s, F, \mathrm{std})$

For Sym^4 Shimura method only using pullback. Needed to develop nearly holomorphic theory for Siegel modular forms.

__Theorem__ (Garrett 1989): $f, g, h \in S_{2k}(1)$

$$\frac{L(4k-2, f \times g \times h)}{\pi^{10k-3} \langle f, f\rangle \langle g, g\rangle \langle h, h\rangle} \in K_f K_g K_h$$

Take Eisenstein series $E(Z, s)$, $Z \in \mathfrak{H}_3 = \{ Z \in \mathrm{Mat}_3(\mathbb{C}) : {}^tZ = Z, \ \mathrm{Im}(Z) > 0 \}$

$$\int_{(SL_2(\mathbb{Z})\backslash\mathfrak{H})^3} E\left(\begin{bmatrix} z_1 & & \\ & z_2 & \\ & & z_3 \end{bmatrix}, s\right) f(z_1)\, g(z_2)\, h(z_3)\, (y_1 y_2 y_3)^{s+1} \, d(\cdot \;)$$

$$= L(s+4k-2,\ f\times g\times h) \times \boxed{}$$

power of π, Γ-factors, ζ-factors.

f, g, h modular forms of level N

$$\int_{\Gamma_0(N)\backslash\mathfrak{H}} f(z)\, g(z)\, h(z)\, \frac{dx\,dy}{y^2} = C \prod_{p} I_p$$

this is by using uniqueness of trilinear forms on representations.

$$= L(1/2,\ f\times g\times h) \prod_{p\mid N} I_p^*$$

Watson
Ichino

RHS: $L(1/2, \Pi)$: Generalized Riemann Hypothesis $L(1/2, \pi) = 0$??

$L(1/2, \pi) \neq 0$, Then $L(1/2, \pi) > 0$?? Yes!! (for a large class by Rallis-Lapid)

Estimates: Convexity bounds $L(1/2, \Pi) \ll N_\pi^{1/2}$

N_π conductor of π

Lindelöf hyp. $L(1/2, \pi) \ll N_\pi^{\varepsilon}$ $\forall\, \varepsilon > 0$

Sub-convexity bound: $L(1/2, \pi) \ll N_\pi^{1/4-\delta}$ $\delta > 0$

Pitale

3-29-15

pg 5

L.H.S: $g = \bar{f}$ $\int h(z) |f(z)|^2 \, d\mu(z)$

$$= \int h(z) \, d\mu_f(z)$$

Question: If f varies over a family with parameter $\to \infty$

<u>Quantum Unique Ergodicity Conj</u>: $d\mu_f \to d\mu$