Krieger
3-28-15
pg1

Unlikely Intersections in Complex dynamics:

I. Unlikely int. in arithmetic geometry.

View $\mathbb{C}$ as a parameter space for isom. classes of ell. curves via j-function.

$$j: \mathbb{H} \to \mathbb{C} \qquad E_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$$
$$\tau \mapsto j(\tau) \qquad E_\tau \cong E_{\tau'} \Leftrightarrow j(\tau) = j(\tau').$$

<u>Thm (Schneider):</u> E_τ has CM $(\mathrm{End}(E_\tau) \supsetneq \mathbb{Z})$. iff τ is imag. quadratic $\Leftrightarrow$ $j(\tau)$ and τ are both algebraic.

<u>Facts about j-invs of CM pts:</u>

⓪ CM j-inv are Zariski dense in $\mathbb{C}$

① Over any given # field K, $\exists$ only finitely many CM j-inv defined $/K$

↳ not a Northcott property.

Consider $\mathbb{C}^2$ as param. pairs of ell. curves.

<u>Def:</u> $(a,b) \in \mathbb{C}^2$ is <u>special</u> if $a = j(\tau_a)$, $b = j(\tau_b)$ and E_{τ_a} & E_{τ_b} both have CM.

<u>Facts about special points:</u>

⓪ Zariski dense in $\mathbb{C}^2$

① Which curves have Zariski dense subset of special points?

$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C} & & \mathbb{C} \end{tikzcd}$$

<u>Type A</u>: Fiber over a CM j-invariant

<u>Type B</u>: image under $j \times j$ of $\{(z, Nz)\}$ where N is a positive integer.

<u>Thm [André]</u> If $C \subset \mathbb{C}^2$ irred. alg. curve containing ∞-ly many special points, then C is of type A or B.

This is part of general Pink-Zilber conjecture.

- Manin-Mumford
- Mordell-Lang, ...

<u>II. Notions and definitions in complex dynamics</u>

Let $f(z) \in \mathbb{C}[z]$; $\deg f > 1$.

<u>Def</u>: $f^n(z) = \overbrace{f \circ \cdots \circ f}^{n\text{-copies}}(z)$. Given $z \in \mathbb{C}$, define the forward orbit of z under f $\mathcal{O}_f(z) = \{f^n(z) : n \geq 1\}$. We say z is <u>preperiodic</u> if $\mathcal{O}_f(z)$ is finite. We say z is <u>periodic</u> if $f^n(z) = z$ for some $n \geq 1$.

<u>Example</u>: $f(z) = z^2$.

$f^2(z) = (z^2)^2 = z^4$

$\vdots$

$f^n(z) = z^{2^n}$

Krieger
3-28-15
p53

So we have z is pre-periodic iff $z^{2^m} = z^{2^n}$ $m \neq n$,
iff z is 0 or a root of unity.

$z \longmapsto \mathcal{O}_f(z) = \{z^2, z^4, \dots\}$.

If $|z| < 1$, then $f^n(z) \to 0$. in modulus.

If $|z| > 1$, then $f^n(z) \to \infty$

If $|z| = 1$, ?

<u>Def:</u> The <u>Fatou set</u> of f is $\mathcal{F}(f) = \{z \in \mathbb{C} : f^n(z)$ forms a normal family $\}$

and the <u>Julia set</u> of f $J(f) = \mathbb{C} \setminus \mathcal{F}$

<u>filled Julia set</u> $K(f) = \{z \in \mathbb{C} : |f^n(z)|$ remains bounded $\}$

$J(f)$ is the boundary of $K(f)$

$f(z) = z^2$ $J(f) = S^1$

$K(f) = \overline{\mathbb{D}}$, $\mathcal{F}(f) = \mathbb{C} \setminus S^1$.

Since $\mu^{-1} \circ f^n \circ \mu = (\mu^{-1} \circ f \circ \mu)^n$ for any $\mu \in \mathrm{Aut}(\mathbb{C})$ we say two dynamical systems are equivalent if they are related by conjugation.

$\mathbb{C}$ can be viewed as a parameter space for classes of quad. poly. dynamical systems.

<u>Fact:</u> $\underset{f_c(z)}{z^2 + c}$ has either $\underset{K(f_c)}{\wedge}$

→ connected : $|f^n(0)|$ remains bounded

→ totally disconnected : $|f^n(0)| \to \infty$.

$c = 0$: connected ✓ (filled unit disk)

Krieger
3-28-15
p. 4

$c = i$: 0 is preperiodic

$c = -1$: 0 is periodic

$c = 1$: $0, 1, 2, 5, 26, \ldots$

Def: The Mandelbrot set is

$M = \{ c \in \mathbb{C} : f_c(z)$ has connected filled Julia set $\}$.

Def: f_c is post-critically finite (PCF) if 0 is preperiodic for f_c. (or c is a post-critically finite point)

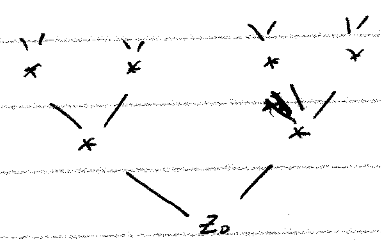
Facts about PCF:

⓪ Zariski dense in $\mathbb{C}$

① all algebraic

② collection of bounded height.

③ Fix $z_0 \in \mathbb{C}$ (don't choose this poorly)



K # field of definition i.e. $c \in K$

$\mathrm{Gal}(\bar{K}/K)$ acts on this tree. T_∞.

$\rho : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Aut}(T_\infty)$

[Jones] In some cases, ρ has finite index image iff f is not PCF.

Conjecture: [Ingram, gen. due to Baker-DeMarco] Consider $\mathbb{C}^2$ as a parameter space $(a, b) \longleftrightarrow (z^2 + a, z^2 + b)$. Call (a, b) special if $z^2 + a$ and $z^2 + b$ are both PCF.

Krieger
3-28-15
pg 5

C contains infinitely many special points iff
① C is a fiber of a projection over PCF point
② $C = \Delta$

Thm [Ghioca-K.-Nguyen-Ye]: Zhang's conjecture is true.

Dictionary between arith. geom. & complex dynamics:

arith. geo of abelian var.:	complex dynamics
CM	PCF
$\mathbb{Z}$ & $\mathbb{Q}$-subgrps gen. by pts	forward and grand orbit
torsion pts	preperiodic pts

1. Some cases of dyn. Manin-Mumford [Ghioca-Tucker-Zhang].
2. Some cases of dyn. Mordell-Lang. [Bell-Ghioca-Tucker]
3. Questions about prime divisors in orbits [Silverman, Zhang, K., ...].