Venkatesh

Spherical Varieties and the Relative Langlands Program:

Joint work w/ Sakellaridis..

1. Integrals of matrix coefficients and the local Plancherel formula
2. Spherical varieties and dual groups
3. Conjectures (local settings) and results.

G reductive algebraic group / F global
$H \subset G$ subgroup.
$\Pi = \bigotimes_v \Pi_v$ auto rep. of $G(\mathbb{A})$.
$\overset{\psi}{\varphi} = \bigotimes_v \varphi_v$.

We would like $\left| \int_{H(F)\backslash H(\mathbb{A})} \varphi \right|^2 = \prod_v |l_v(\varphi_v)|^2$ $\quad l_v$ = local functionals on Π_v inv. under $H(F_v)$

Example:

Waldspurger, Ichino & Ikeda

$$G = SO_n \times SO_{n+1} \qquad \pi \text{ tempered}, \subset L^2_{disc}$$

$$H = SO_n$$

$$\left| \int_{H(F)\backslash H(\mathbb{A})} \varphi \right|^2 \text{ "="} \underbrace{c}_{\text{power of } 2} \prod_v \int_{H(F_v)} \langle h\varphi_v, \varphi_v \rangle \, dh$$

Remarks: $\int \langle h\varphi_v, \varphi_v \rangle \geq 0$

By uniqueness of an $H(F_v)$-inv. functional on π_v,

$$\int_{H(F_v)} \langle h\varphi_v, \varphi_v \rangle = |l_v(\varphi_v)|^2 \text{ where}$$

l_v is a $H(F_v)$-inv. linear form, unique up to $z \in \mathbb{C}^\times$, $|z| = 1$.

<u>Conjecture (Ichino-Ikeda):</u> If $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$, then

$$\int_{H(F_v)} \langle h\varphi_v, \varphi_v \rangle \neq 0 \text{ for some } \varphi_v.$$

(Proved Sakellaridis - V.)

<u>§1:</u> G = reductive alg. grp. / k local

H = subgroup We make the following assumptions

- For every tempered unitary rep. σ, $\int \langle hv, v\rangle_\sigma < \infty$.
- (not important) $\dim \mathrm{Hom}_H(\sigma, \mathbb{C}) \leq 1$.

For any tempered σ of $G(k)$,

$$\int_{H(k)} \langle hv, v\rangle = |l_\sigma(v)|^2$$

normalizes $l_\sigma : \sigma \to \mathbb{C}$.

By Frobenius reciprocity; l_σ extends

$$L_\sigma : \sigma \hookrightarrow C\left(G(k)/H(k)\right)$$

Take adjoint,

$$L_\sigma^* : C_c(G/H) \longrightarrow \sigma$$

<u>Fact:</u> For $F \in C_c(G/H)$,

$$(*) \qquad \|F\|^2_{L^2(G/H)} = \int_{\sigma \in \hat{G}} d\mu_{pl}(\sigma)\, |L^*_\sigma F|^2 .$$

planchercl measure for $G(k)$

<u>Question:</u> understand $L^2(G(k)/H(k))$ with hope

Plancherel formula

$\updownarrow$

local factors for global periods.

G split reductive group / k local

A <u>spherical variety</u> under G is homogeneous ($X = G/H$) variety such that $B \subset G$ (B = Borel) acts with Zariski open orbit.

<u>Examples:</u> If G = semisimple, H any symmetric subgroup, then G/H is spherical.

a) $G = GL_{2n}$ $\quad H = Sp_{2n}$

b) $G = GL_{2n}$ $\quad H = GL_n \times GL_n$

c) $G = GL_n$ $\quad H = O_n$.

d) H any s.s. group. $G = H \times H$, $H \curvearrowright G$ diagonally.

"group case" $\quad G/H = H$

left $\curvearrowright$ $\curvearrowleft$ right
$H \times H$

$G = SO_n \times SO_{n+1}$ $\quad H = SO_n$

G = any, H = max. unipotent.

Miracle: (Losev)

Spherical varieties can be classified by combinatorial data.

Dual groups:

Knop attached to (G, X) a root system.

Gaitsgory & Nadler: corresponding reductive group $\underset{\hat{G}_X}{\underbrace{}} \hookrightarrow \hat{G}$.

Sak-V.: construct an embedding

$$\rho_X : SL(2) \longrightarrow \hat{G}$$

commuting with $\hat{G}_X$.

In case a) above: $\hat{G} = GL_{2n}(\mathbb{C})$

$$\hat{G}_X = \underset{A}{GL_n(\mathbb{C})} \hookrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$\rho_X : \underset{SL(2)}{\text{nontrivial commuting}} \quad \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto \begin{pmatrix} t & & & & & 0 \\ & \ddots & & & & \\ & & t & & & \\ & & & t^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & t^{-1} \end{pmatrix}.$$

In case b) above: $\hat{G} = GL_{2n}(\mathbb{C})$

$$\hat{G}_X = Sp_{2n} \qquad \rho_X \text{ trivial.}$$

Conjecture (Sak.-V.): For unitary rep. π of $G(k)$ to occur in $L^2(X(k))$, π must belong to an Arthur packet A_ψ where parameter

$$\psi : WD_k \times SL(2) \longrightarrow \hat{G}$$

which is X-distinguished →

$$\begin{tikzcd} WD_k \arrow[d] & \times \; SL(2) \arrow[r, "\rho_X"] & \hat{G} \\ \hat{G}_X \arrow[urr] & & \end{tikzcd}$$

Verified for some cases where rank $\hat{G}_X$ small by Gan & Gomez.

We reduce this to the case of "discrete series" $(L^2_{disc}(X))$
(In fact, under assumptions on X, give Pl. formula given L^2_{disc}.)

The story we try to imitate is:

$$\underset{\substack{\text{parabolic}\\ \text{Ind}}}{MN} \xrightarrow{\text{IND}} G$$

$$\underset{\text{Levi}}{\hat{M}} \hookrightarrow \hat{G}$$

Starting from parabolic subgroups

$$\hat{P} \subset \hat{G}_X$$

one can produce a spherical G-variety Y whose dual group is Levi of $\hat{P}$; $\hat{G}_Y = \hat{M} = \mathrm{Levi}(\hat{P})$.

$$\begin{array}{ccc} Y & & X \\ \hat{M} & \subset & \hat{G}_X \end{array}$$

We construct : $L^2(Y(k)) \longrightarrow L^2(X(k))$

(k nonarch) $C_c^\infty(Y(k)) \longrightarrow C_c^\infty(X(k))$.

If $\hat{P}, \hat{Q}$ have conjugate Levi subgroups,

$$\begin{tikzcd} L^2(Y_{\hat{P}}) \arrow[d, "\wr"] \arrow[dr] & \\ L^2(Y_{\hat{Q}}) \arrow[r] & L^2(X) \end{tikzcd}$$

Reduce $L^2(X)$ to $L^2_{disc}(Y_{\hat{P}})$

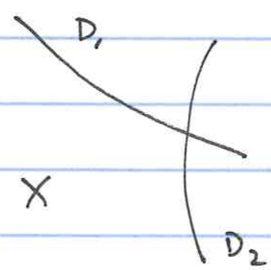
<u>What are $Y_{\hat{P}}$?</u>

Assume $\mathrm{Aut}_G(X) = $ trivial $\Leftrightarrow N(H) = H$

(not important, but makes statements easier)

X has canonical G-eqvt compactification "wonderful compactification" $\bar{X}$

$\bar{X} - X$ = union of normal crossing divisors. $\longleftrightarrow$ simple roots of $\hat{G}_X$.



If $\hat{P}_\alpha$ = max. parabolic obtained by deleting α, then $Y_{\hat{P}_\alpha}$ = normal bundle to D_α in $\bar{X}$.

<u>(Speculative) global conjecture.</u>

~~Normaliza~~ A sharper form of conjecture normalizes a Pl. measure for $L^2(X(k))$. $\longleftrightarrow$ normalizing a H-invt. linear fctnl. $\sigma \to \mathbb{C}$ on any unitary.

Venkatesh
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The hope is this is the correct normalization to do global periods.