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Automorphic forms and Galois representations:

F number field
ℓ rational prime
$\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$

Def: ① A cuspidal autom. rep. π of $GL_n(/\mathbb{A}_F)$ is algebraic if the Harish-Chandra parameters of ∞-char of π_∞ $HC(\pi_\infty)$ lies in $\left(\mathbb{Z}^n/S_n\right)^{Hom(F,\mathbb{C})} \subset \left(\mathbb{C}^n/S_n\right)^{Hom(F,\mathbb{C})}$.

② If $r_\ell : Gal(\overline{F}/F) = G_F \to GL_n(\overline{\mathbb{Q}}_\ell)$ is a cont. s.s. rep. (ℓ-adic rep.) we call r algebraic if

a) r is unramified a.e.

b) $\forall v|\ell$, $r|_{G_{F_v}}$ is deRham

$$\rightsquigarrow \quad HT(r) = \left(\mathbb{Z}^n/S_n\right)^{Hom(F,\overline{\mathbb{Q}}_\ell)} : \forall \tau : F \hookrightarrow \overline{\mathbb{Q}}_\ell, \quad F \hookrightarrow F_{v_\tau} \to \overline{\mathbb{Q}}_\ell$$

$$\dim_{\overline{\mathbb{Q}}_\ell}\left(r \otimes_{\tau, F_{v_\tau}} \widehat{\overline{F}}_v(\text{cyclo}^j)\right)^{G_{F_v}} = \text{mult. of } j \text{ in } HT(r).$$

Example: X/F smooth proj. variety.

$r = H^i(X \times \overline{F}, \overline{\mathbb{Q}}_\ell)$ is algebraic

mult. of j in $HT(r)_\tau = \dim_{\mathbb{C}} H^{j,i-j}(X \times_{F,\tau} \mathbb{C}, \mathbb{C})$.

Conjecture: (Langlands-Clozel-Fontaine-Mazur) There is a bijection

$$\left\{\begin{array}{c}\text{alg. cusp. auto} \\ \text{reps of } GL_n(/\mathbb{A}_F)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irred alg. } \ell\text{-adic reps} \\ G_F \to GL_n(\overline{\mathbb{Q}}_\ell)\end{array}\right\}$$

$$\pi \longmapsto r_\ell(\pi)$$

s.t. 1) $HC(\pi_\infty) = HT(r_\ell(\pi_\infty))$.

2) $\forall$ prime v of F,

$$WD(r_\ell(\pi)|_{G_{F_v}})^{\text{Frob-s.s.}} \simeq \mathrm{rec}_{F_v}(\pi_v)$$

repos. of $W_{F_v} \times SL_2$, $0 \to I_{F_v} \to W_{F_v} \to \mathrm{Frob}_v^{\mathbb{Z}} \to 0$

- rec_{F_v} local Langlands $\qquad \sigma \mapsto \mathrm{Frob}_v^{v(\sigma)}$.
- Frob-s.s. $\leftrightarrow$ semi-simplify

$$Sp_2 : W_{F_v} \longrightarrow SL_2(\mathbb{C}) \text{ indecomposable}$$

$$\sigma \longmapsto \begin{pmatrix} \#k_v^{-\frac{1}{2}v(\sigma)} & * \\ 0 & \#k_v^{\frac{1}{2}v(\sigma)} \end{pmatrix}$$

$$(id, Sp_2) = j_v : W_{F_v} \longrightarrow W_{F_v} \times SL_2(\mathbb{C})$$

$v \nmid \ell \qquad r_\ell(\pi)|_{W_{F_v}} \simeq WD(r_\ell(\pi)|_{G_{F_v}}) \circ j_v$.

$v | \ell$ see Fontaine.

$n=1$ this case is true. This is essentially class field theory.

Past $n=1$ one must impose severe restrictions to get any theorems.

① <u>Regularity</u>

π is called <u>regular</u> if $HC(\pi_\infty)_\tau$ consists of n distinct integers $\forall \tau$.

r is called <u>regular</u> if $HT(r)_\tau$ consists of n distinct integers $\forall \tau$.

② polarizable

π is polarizable if

1) F is CM or totally real, F^+ max. tot. real subfield.
2) $\exists\ \chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \longrightarrow \mathbb{C}$ s.t.

$$\pi^c \simeq \pi^\vee \otimes (\chi \circ N_{F/F^+} \circ \det)$$

and $\chi_v(-1)$ is indep. of $v \mid \infty$.

r is polarizable if · F tot. real, $r : G_F \to GO_n(\overline{\mathbb{Q}}_\ell)$
multiplier totally even
$\text{mult}(r)(c_v) = 1\ \ \forall\, v \mid \infty$

or

· F is totally real, $r : G_F \longrightarrow GSp_n(\overline{\mathbb{Q}}_\ell)$
multiplier totally odd
$\text{mult}(r)(c_v) = -1\ \ \forall\, v \mid \infty$.

or

· F is imaginary, $\exists$ a symm. pairing $\langle\, ,\rangle$ s.t.
$\langle r(\sigma)x, r(~~conj~~\, c\sigma c)y\rangle = \chi(\sigma)\langle x, y\rangle$
where $\chi : G_{F^+} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$
$\chi(c_v)$ indep. of $v \mid \infty$.

Theorem: If π is a polarizable, reg, alg. cusp. auto. rep. of $GL_n(\mathbb{A}_F)$, then $\exists$ a polarizable, reg. alg. ℓ-adic rep. $r_\ell(\pi) : G_F \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$ s.t.

1) $HT(r_\ell(\pi)) = HC(\pi_\infty)$
2) $WD(r_\ell(\pi)|_{G_{F_v}})^{F\text{-}s.s} = \text{rec}_{F_v}(\pi_v)\ \ \forall v$

Moreover π_v is tempered $\forall v$.

This theorem is due to Shin, Chenevier-Harris, Caraiani, ...

shin regular case: $r_\ell(\pi) \subset$ cohom. of unitary group Shimura variety $\leftrightarrow G$

$G(\mathbb{R}) \cong G(U(n-1,1) \times U(n)^?)$ n odd

$G(U(n,1) \times U(n+1)^?)$ n even.

other cases: $r_\ell(\pi)$ is an ℓ-adic limit

and $r_\ell(\pi)^{\otimes 2} \longrightarrow G(U(n-1,1)^2 \times U(n)^?)$

<u>Theorem:</u> Suppose r is a polarizable regular alg. ℓ-adic rep. $r: G_F \to GL_n(\overline{\mathbb{Q}}_\ell)$. Assume further

a) $\ell \geq 2(n+1)$, $e^{2\pi i/\ell} \notin F$.

b) $\bar{r} = (r \bmod \ell)|_{G_{F(\zeta_\ell)}}$ irreducible

c) ℓ unramified in F, $r|_{G_v}$ is crystalline $\forall v | \ell$ and $HT(r) \in ((\mathbb{Z} \cap [0, \ell-2])^n / S_n)^{\mathrm{Hom}(F, \overline{\mathbb{Q}}_\ell)}$

then $\exists$ a finite Galois CM extension F'/F and a polarizable, reg, alg. cusp auto rep. π' of $GL_n(\mathbb{A}_{F'})$ with $r|_{G_{F'}} \cong r_\ell(\pi')$.

Conditions a) - c) are true for most ℓ in an irred. family.

Theorem due to Barnet-Lamb, Gee, Geraghty, T.

Note one requires a finite base change F'/F. However, for applications this is usually enough.

$\Rightarrow$ r pure

$\Rightarrow$ r is part of a family as l-varies

$\Rightarrow$ $L(r,s) = \prod_v \det(1 - r^{I_v}(\mathrm{Frob}_v)\, \# k_v^{-s})^{-1}$ converges in some right half-plane, has meromorphic cont. to $\mathbb{C}$, and satisfies expected functional equation.

$\Rightarrow$ $\mathrm{Sym}^{n-1}(H^1(E, \bar{\mathbb{Q}}_l))$

$E/\mathbb{Q}$ ellip. curve

$$\Rightarrow \frac{1+p-\# E(\mathbb{F}_p)}{\sqrt{p}} \in [-2,2]$$

equidistributed wrt $\frac{1}{2\pi}\sqrt{4-t^2}\,dt$ (Sato-Tate)

On the Galois side regular seems very restrictive, but on the automorphic side it seems to be very common.

<u>Theorem:</u> (Harris, Lan, T., Thorne) Suppose F is CM or tot. real, and π is a regular alg. cusp. auto. rep. of $GL_n(/\mathbb{A}_F)$. Then $\exists$ cts. l-adic reps.

$$r_l(\pi) : G_F \longrightarrow GL_n(\bar{\mathbb{Q}}_l)$$

s.t. for all but finitely many v

$$WD(r_l(\pi)|_{G_{F_v}})^{F\text{-}s.s} = \mathrm{rec}_{F_v}(\pi_v).$$

(This is not constructed in coh. of Shim. variety, but as limit. It is believed they do not exist in cohom. of Shim. variety!)