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Stability and the Local Langlands Correspondence:

k = number field, v place of k

$\rho: \underset{\overset{\shortparallel}{G}}{Gal(\bar{k}/k)} \longrightarrow Aut(V) = GL_n(\mathbb{C})$

Artin: $\rho_v = \rho|_{G_v}$.

$I_v \subseteq G_v$ $\qquad L(s, \rho_v) = \det(1 - \rho(Frob_v)|V^{I_v} q_v^{-s})^{-1}$

$L(s, \rho) = \prod_{v<\infty} L(s, \rho_v) \; L_\infty(s, \rho)$

$\varepsilon(s, \rho) = W(\rho)(|d_{k/\mathbb{Q}}|^n N(f(\rho)))^{-s+1/2}$

$L(s, \rho) = \varepsilon(s, \rho) L(1-s, \rho^\vee)$.

L is defined locally. What can be said about ε?

Dwork ('50's), Langlands ('60's), Deligne (1972): $\varepsilon(s,\rho) = \prod_v \varepsilon(s, \rho_v, \psi_v)$

$1 \neq \psi \in (k \backslash \mathbb{A}_k)^\vee$.

Let ρ_1 and ρ_2 be n-dim cont. rep. of $G := Gal(\bar{F}/F)$.

F = local p-adic $\qquad \psi = \psi_F$

χ char. of $\hat{F}^\times$, highly ramified.

$\det(\rho_1) = \det(\rho_2)$

Stability: Suppose ρ_1 and ρ_2 as before. Then

$$\varepsilon(s, \rho_1 \otimes \chi, \psi) = \varepsilon(s, \rho_2 \otimes \chi, \psi).$$

L.L.C = Local Langlands correspondence. n and m dim resp.

ρ_1, ρ_2 reps of W_F (= Weil group) continuous complex.

Harris-Taylor: $\pi(\rho_1), \pi(\rho_2)$ = irred. adm. rep. of $GL_n(F)$, $GL_m(F)$ resp.

$$\varepsilon(s, \rho_1 \otimes \rho_2, \psi) = \varepsilon(s, \pi(\rho_1) \otimes \pi(\rho_2), \psi)$$
$$L(s, \rho_1 \otimes \rho_2, \psi) = L(s, \pi(\rho_1) \otimes \pi(\rho_2), \psi).$$

Act

$$\gamma(s, \rho, \psi) = \varepsilon(s, \rho, \psi) \frac{L(1-s, \rho^\vee)}{L(s, \rho)}$$

$$\gamma(s, \pi(\rho), R, \psi) = \varepsilon(s, \pi(\rho), R, \psi) \frac{L(1-s, \pi(\rho), \check{R})}{L(s, \pi(\rho), R)}$$

R = complex rep. of $GL_n(\mathbb{C})$.
(Λ^2, Sym^2 for example)

Stability & Functoriality:

G = connected reductive group $/k$, quasi split.
${}^L G = \hat{G} \rtimes W_k$ $\qquad {}^L G \hookrightarrow GL_N(\mathbb{C}) \times W_k$.
Assume $\hat{G}_D$ = classical group
Contains classical groups (including unitary groups) and GSpin.

$\pi = \otimes_v \pi_v$ = cuspidal auto. rep. of $G(\mathbb{A}_k)$, generic.
$\sigma = \otimes_v \sigma_v$ = cuspidal auto rep. of $GL_m(\mathbb{A}_k)$.
$L(s, \pi_v \times \sigma_v)$, $\varepsilon(s, \pi_v \times \sigma_v, \psi_v)$ are defined.
$GL_m \times G = M$ = maximal Levi in a bigger $\tilde{G}$.

Langlands. conj. $\pi \rightsquigarrow \Pi = \otimes_v \Pi_v$ = auto.

$$\Pi = \bigotimes_{v\in S} \Pi_v \bigotimes_{v\notin S} \Pi_v \qquad \Pi' = \bigotimes_{v\in S} \Pi_v' \bigotimes_{v\notin S} \Pi_v' \text{ = auto rep.}$$

w/

$$\Pi_v' = \Pi_v.$$

One needs converse theorem to get this. One wants

$$\gamma(s, \pi_v \times \sigma_v, \psi_v) = \gamma(s, \Pi_v' \times \sigma_v, \psi_v).$$

$\sigma_v = \sigma_v^\circ \otimes \chi_v$ Ask

(*) $\gamma(s, \pi_v \times \sigma_v^\circ \otimes \chi_v, \psi_v) = \gamma(s, \Pi_v' \times \sigma_v \otimes \chi_v, \psi_v)$

Then (*) holds if χ is highly ramified.

$$\gamma(s, \pi_v \times \sigma_v^\circ \otimes \chi_v, \psi_v) = \prod_{i=1}^{m} \gamma(s, \pi_v \otimes \chi_{i,v}, \psi_v) \qquad \chi_{i,v} \in \hat{k}_v^\times.$$

Prove (*) for $G \times GL_1$. Assume ω_{π_v} & $\omega_{\pi_v'}$, χ_v highly ram.

They prove (Cogdell-PS)

$$SO(2n+1): \gamma(s, \pi_v \otimes \chi, \psi_v) = \gamma(s, \pi_v' \otimes \chi_v, \psi_v).$$

Transfer from $SO(2n+1)$ to $GL(2n)$ already done (IHES 2001)

(G, M) = G, M pair. $P = MN$ $M \supset T$, $N \subset U$.

P maximal

γ factors were defined by means of local coefficients.

π irred. gen. rep. of $M(F)$

partial Bessel fctn.

$$C_\psi(s, \pi)^{-1} \sim \int \text{PBF}(m)\,\text{char}(m)\,dm$$

$Z_M \backslash M_M^N$

char. fctn of some compact open subgrp of $\bar{N} = w_0 N^- w_0^{-1}$

$w_0 = w_{\ell,G} w_{\ell,M}^{-1}$.

N^- = opposite of N

Get it for all $G \times GL_1$

$(GSp_{2n},\ GL_n \times GL_1$ $\qquad C_\psi(s,\pi) = \gamma(s,\pi,st,\psi)\gamma(2s,\pi,\Lambda^2,\psi).$

π irred adm gen.

Jacquet-Ye: germ expansion for the full Bessel fctn for GL_n when π is supercuspidal.

$$\int_{\text{stab.}\backslash U_m \times U_m} f(u_1 m u_2)\,\psi(u_1)\,\psi(u_2)\,du_1\,du_2 \qquad f \in C_c^\infty(GL_n(F)).$$

This is for full Bessel function. Still need it for partial Bessel fctn.

LLC: $\rho \longleftrightarrow \pi(\rho)$

$R = \Lambda^2,\ Sym^2$

$$\gamma(s, R\cdot\rho, \psi) \overset{?}{=} \gamma(s, \pi(\rho), R, \psi).$$

If you know this for Λ^2, you have it for Sym^2 b/c

$$\rho \otimes \rho = \Lambda^2 \oplus Sym^2.$$

stability we use.

$R = \Lambda^2$: $\gamma(s,\pi(\rho_2)\otimes\chi,\Lambda^2,\psi) = \gamma(s,\pi(\rho_1)\otimes\chi,\Lambda^2,\psi)$ $\qquad$ χ high. ram. ρ = irred.

$\det\rho_1 = \det\rho_2$

Prop. $n \in \mathbb{N}$, ρ = n-dim $\mathbb{C}$-rep. of W_F. Then for all χ = high. ram. char.

(stable version)

$$\gamma(s, \Lambda^2(\rho\otimes\chi), \psi) = \gamma(s, \pi(\rho)\otimes\chi, \Lambda^2, \psi).$$

Lemma 2: $n \in \mathbb{N}$, $\omega_0 \in \hat{F}^\times \Rightarrow \exists\, k$ = # field,

$\exists\ \tilde{\rho}$ = irred. rep. of W_k s.t. $\tilde{\rho}_w = \tilde{\rho}|_{W_{k_w}}$

1) $\exists\, v$ place of k. s.t. $k_v = F$, $\tilde{\rho}_v$ = irred.

$\det \tilde{\rho}_v = \omega_0$

2) $\forall\, w \neq v$, $w < \infty$, $\tilde{\rho}_w$ = reducible

3) $\pi(\tilde{\rho}) := \bigotimes_v \pi(\tilde{\rho}_v)$ cuspidal auto.

$\tilde{\chi} = \bigotimes_w \tilde{\chi}_w$ h.r. at all the bad places.

compare fctn. eqn. for $\pi(\tilde{\rho})$ and $\tilde{\rho}$ by induction on n.

$$\gamma(s, \Lambda^2(\tilde{\rho}_w \otimes \chi), \psi) \equiv \gamma(s, \pi(\tilde{\rho}_w) \otimes \chi, \Lambda^2, \psi) \qquad w \neq v.$$

$\rho_0 = \tilde{\rho}_v \qquad \pi_0 = \pi(\rho_0)$

ρ ∮ $\det \rho = \det \rho_0 = \omega_0$

$$\gamma(s, \Lambda^2(\rho_0 \otimes \chi), \psi) = \gamma(s, \pi(\rho_0) \otimes \chi, \Lambda^2, \psi)$$

Deligne $\Big\}\ \|$ $\qquad\qquad \Big\} = ?$

$$\gamma(s, \Lambda^2(\rho \otimes \chi), \psi) = \gamma(s, \pi(\rho) \otimes \chi, \Lambda^2, \psi).$$

Prove the full equality for a basis of Grothendieck group of W_F.

Brauer's thm $\Rightarrow$ choose monomials

$\tilde{\rho}$ = rep. of W_k, $k_v = F$ $\qquad \pi(\tilde{\rho})$ exists $= \bigotimes_w \pi(\rho_w)$
monomial

$\tilde{\chi} = \bigotimes_w \tilde{\chi}_w \qquad w \neq v$ bad prime, h.r.

$\chi_v = 1 \qquad \chi_w = 1 \qquad w = \infty$

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$$\gamma(s, \Lambda^2(\bar{\rho}_w \otimes \tilde{\chi}_w), \psi) = \gamma(s, \pi(\bar{\rho}_w) \otimes \tilde{\chi}_w, \Lambda^2, \psi) \quad w \neq v$$