Converse Theorems:

Integral Representations for $GL_n \times GL_m$, $m<n$:

$$GL_n \supseteq P_n = \left\{ \begin{pmatrix} * & * \\ 0 \dots 0 & 1 \end{pmatrix} \right\} \supseteq N_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

mirabolic

$$\cup\!\!|$$

$$Y_{n,m} \left\{ \left(\begin{array}{c|c} I_{m+1} & * \\ \hline 0 & z \end{array} \right) \middle| \, z \in N_{n-m+1} \right\}$$

k = global field

$\psi : k \backslash \mathbb{A} \to \mathbb{C}^1$

$\psi(n) = \psi(\sum n_{i,i+1})$

$\pi = \otimes \pi_v$ cusp. rep. $GL_n(\mathbb{A})$, $\pi' = \otimes \pi'_v$ cusp. rep. $GL_m(\mathbb{A})$

if $\varphi \in V_\pi$, $\varphi' \in V_{\pi'}$

$$\mathbb{P}_m \varphi(p) = |\det p|^{\frac{-n-m-1}{2}} \int_{Y_m(k) \backslash Y_m(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & \\ & I_{n-m-1} \end{pmatrix} \right) \psi(y)\, dy$$

$p \in P_{m+1}$

cuspidal fctn on $P_{m+1}(\mathbb{A})$.

$$I(s, \varphi, \varphi') = \int_{GL_m(k) \backslash GL_m(\mathbb{A})} \mathbb{P}_m \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) \, |\det h|^{s-1/2} \, dh$$

$$= \prod I_v(s, W_{\varphi,v}, W_{\varphi',v}) \qquad Re(s) \gg 0$$

$$= \left(\prod_{v \in S} \frac{I_v(s, W_{\varphi,v}, W_{\varphi',v})}{L(s, \pi_v \times \pi'_v)} \right) L(s, \pi \times \pi').$$

Thm (J.-PS-S): $L(s,\pi\times\pi') = \prod_v L(s,\pi_v\times\pi_v')$

is "nice"

- entire continuation
- bounded in vertical strips
- $L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\tilde{\pi}\times\tilde{\pi}')$.

Converse theorem inverts this.

Take $\pi = \otimes\pi_v$ irred. adm. rep. of $GL_n(\mathbb{A})$. π encodes an Euler product

$$L(s,\pi) = \prod L(s,\pi_v), \quad Re(s)>0.$$

The question is when is π automorphic?

$$\mathcal{T}(m) = \coprod_{1\le d\le m} \{\pi' = \otimes\pi_v' : \text{cusp. auto. rep. of } GL_d(\mathbb{A})\}.$$

For any $\pi'\in\mathcal{T}(m)$ we can form

$$L(s,\pi\times\pi') = \prod L(s,\pi_v\times\pi_v') \qquad Re(s)>>0.$$

Question: If $L(s,\pi\times\pi')$ is nice $\forall\ \pi'\in\mathcal{T}\subset\mathcal{T}(n-1)$, is π (cuspidal), automorphic?

Ilya's results fall into 3 families: (determining $\mathcal{T}$).

1) Restrict rank (spectral inversion)
2) Restrict ramification (spectral inversion & generation of congruence subgroups)

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3) GL(1) ~~twist~~ twist.

Restricting Rank:

Theorem: If $\mathcal{T} = \mathcal{T}(n-1)$, then π is cuspidal automorphic.

$$\text{If } \xi \in V_\pi \rightsquigarrow W_\varphi(g) \longmapsto U_\xi(g) = \sum_{p \in N_n(k) \backslash P_n(k)} W_\xi(pg)$$

has this even
though not auto..
can make it generic

$$V_\xi(g) = \sum_{q \in N_n'(k) \backslash Q_n(k)} W_\xi(\alpha_n q g)$$

$Q_n =$ opposite mirabolic $= \left\{ \begin{pmatrix} * & 0 \\ * & \vdots \\ & 0 \\ & 1 \end{pmatrix} \right\}$, $\alpha_n = \begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \\ & \vdots \end{pmatrix}$

Use these to form

$$I(s, U_\xi, \varphi') = \left(\prod_{v \in S} \frac{I(s, W_v, W_v')}{L(s, \pi_v \times \pi_v')} \right) L(s, \pi \times \pi') \qquad Re(s) > 0$$

$$I(s, V_\xi, \varphi') = \left(\prod_{v \in S} \quad '' \quad \right) L(1-s, \tilde{\pi} \times \tilde{\pi}') \qquad Re(s) \ll 0$$

The assumptions then allow one to conclude

$$I(s, U_\xi, \varphi') = I(s, V_\xi, \varphi')$$

$\Rightarrow$

$$U_\xi(h_1) = V_\xi(h_1) \Rightarrow U_\xi(g) = V_\xi(g)$$

$$\Rightarrow \xi \mapsto U_\xi \text{ embeds } V_\pi \hookrightarrow \mathcal{A}(GL_n(k)\backslash GL_n(\mathbb{A})).$$

<u>Useful variant</u>: Fix T a finite set of finite places.

$$\mathcal{T}^T(n-1) = \{\pi' \in \mathcal{T}(n-1) : \pi'_v \text{ is unramified } \forall v \in T\}$$

If $L(s, \pi \times \pi')$ is nice $\forall \pi' \in \mathcal{T}^T(n-1)$, then π is quasi-automorphic; $\exists$ an automorphic Π s.t.

$$\Pi_v \simeq \pi_v \quad \forall v \notin T.$$

<u>Thm</u>: If $\mathcal{T} = \mathcal{T}(n-2)$ or $\mathcal{T}^T(n-2)$, the conclusions are exactly the same.

When you run the spectral inversion arg., you conclude that

$$\mathbb{P}_{n-2} U_\xi(h_1) = \mathbb{P}_{n-2} V_\xi(h_1).$$

Recall $\mathbb{P}_{n-2}$ involved integration over

$$Y_{n,n-2} = \left\{ \left(\begin{array}{c|c} I_{n-1} & \begin{matrix} x \\ \vdots \\ x \end{matrix} \\ \hline 0 & 1 \end{array} \right) \right\}$$ against a character.

From here he uses a clever local condition to get the result.

What about $\mathcal{T}(n-3)$?

Now you have to Fourier analyze integral over

$$Y_{n,n-3} = \left\{ \left(\begin{array}{c|cc} I_{n-2} & \begin{matrix} x \\ \vdots \\ x \end{matrix} & \begin{matrix} x \\ \vdots \\ x \end{matrix} \\ \hline 0 & \begin{matrix} 1 \\ 0 \end{matrix} & \begin{matrix} x \\ 1 \end{matrix} \end{array} \right) \right\}$$

This is now a nonabelian problem!

All applications of these theorems (functoriality) were of the theorems for $\mathcal{T} = \mathcal{T}^T(n-1)$.

The strongest expected is the following conjecture:

<u>Conjecture (Jacquet)</u>: Same results $\mathcal{T}([\frac{n}{2}])$ or $\mathcal{T}^T([\frac{n}{2}])$.

Restricting Ramification:

Take S a finite set of places, $S \supset S_\infty$, the ring of S-integers $\mathcal{O}_S$ has class number 1.

$$\mathcal{T}_S(n-1) = \{\pi' \in \mathcal{T}(n-1) : \pi'_v \text{ is unramified } \forall v \notin S\}.$$

<u>Theorem</u>: Suppose $n \geq 3$ and $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_S(n-1)$. Then $\exists$ an auto. rep. Π s.t. $\Pi_v \simeq \pi_v$ $\forall v \notin S$ and for all $v \notin S$ when π_v is unramified.

Here one compensates for restriction by

1) Theory of the conductor for GL_n
2) Generation of congruence subgroups.

$\forall\, v \notin S$, take $\xi_v^\circ \in V_{\pi_v}$ the essential new vector

Fixed by

$$K_1(\mathfrak{p}_v^{n_v}) = \left\{ g \in GL_n(\mathcal{O}_v) : g \equiv \begin{pmatrix} * & * \\ 0 \cdots 0 & 1 \end{pmatrix} \bmod \mathfrak{p}_v^{n_v} \right\}.$$

$\mathfrak{m} = \prod_{v \notin S} \mathfrak{p}_v^{n_v} \subset \mathcal{O}_S$. $\qquad \xi \in V_{\pi_S} = \bigotimes_{v \in S} V_{\pi_v}$

$G_S = \prod_{v \in S} GL_n(k_v)$. For $\xi \otimes \xi^\circ$, $\xi^\circ = \bigotimes_{v \notin S} \xi_v^\circ$,

$U_{\xi \otimes \xi^\circ}$, $V_{\xi \otimes \xi^\circ}$ ... do spectral inversion.

$$V_{\xi \otimes \xi^\circ}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = U_{\xi \otimes \xi^\circ}\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$$

View U_ξ, V_ξ as functions on G_S. U_ξ is still invariant under $P_n(\mathcal{O}_S)$. V_ξ is still invariant under $Q_n(\mathfrak{m})$.

__Theorem:__ $\langle P_n(\mathcal{O}_S), Q_n(\mathfrak{m}) \rangle = \Gamma_1(\mathfrak{m}) \subset G_S$.

$$\xi \longmapsto U_\xi$$
$$V_{\pi_S} \hookrightarrow \mathcal{A}(\Gamma_1(\mathfrak{m}) \backslash G_S).$$

Now use strong approximation

$$V_\pi \dashrightarrow \mathcal{A}(GL_n(k) \backslash GL_n(\mathbb{A})).$$

might have to move
at a finite # of places

Again, it seems this converse theorem has never been used.

GL(1) Twists:

Conjecture: $\pi = \otimes \pi_v$ irred. adm. rep. of $GL_n(\mathbb{A})$. It encodes an Euler product

$$L(s,\pi) = L(s) = \prod_v L(s, \pi_v) \qquad \text{degree } n/k.$$

Assume $L(s)$ converges $\mathrm{Re}(s) > 0$. Suppose $L(s, \pi \times \omega)$ is nice $\forall\, \omega \in \mathcal{J}(1)$ (idele class char.) Then $\exists$ an auto rep. $\Pi = \otimes_v \Pi_v$ s.t.

$$L(s, \Pi_v \times \omega_v) = L(s, \pi_v \times \omega_v)$$

for all v where π_v ~~is~~, ω_v are unramified.

The statement here is really about the Euler product, namely, the Euler product $L(s)$ is automorphic. It is not so much a statement about π. This is more in the spirit of Hecke's converse thm..

The feeling is the most arithmetic applications will come from such a GL(1) converse theorem. The converse for GL(4) w/ GL(1) twist in terms of π being automorphic is not true, so this conjecture is what PS believed was really true.