Arthur
4-23-12
pg 1

The Endoscopic Classification of Representations:

1: Classification Theorem:

F global, char 0.

1) $G = SO(2m+1)$ splits, $\hat{G} = Sp(2m, \mathbb{C}) = {}^LG$ (B_m)

2) $G = Sp(2m)$ splits, $\hat{G} = SO(2m+1) = {}^LG$ (C_m)

3) $G = SO(2m)$ quasi split, $\hat{G} = SO(2m, \mathbb{C})$ (D_m)
$${}^LG = \hat{G} \rtimes Gal(E/F), \quad deg(E/F) = 1, 2.$$

L_F = (hypothetical) global Langlands group.
- locally compact extension $L_F \to W_F$ ← global Weil group.

whose N-dimensional unitary irred. reps. parameterize unitary cuspidal auto reps. of $GL(N)$; with conjugacy classes of local embeddings $L_{F_v} \subset L_F$ where the local Langlands group
$$L_{F_v} = \begin{cases} W_{F_v} & v \text{ arch.} \\ W_{F_v} \times SU(2) & v \text{ p-adic.} \end{cases}$$

Write $\Psi(G)$ = equivalence classes of "L-homomorphisms"
$$\psi : L_F \times SU(2) \longrightarrow {}^LG, \quad im(\psi) \text{ bounded (rel. compact)}$$
(if G were $GL(n)$, this would be saying unitary) taken up to $\hat{G}$-conjugation. Given $\psi \in \Psi(G)$,
$$S_\psi = Cent(\{im(\psi)\}, \hat{G})$$
$$\mathcal{S}_\psi = S_\psi / Z(\hat{G})^\Gamma S_\psi^\circ \quad \text{- finite abelian 2-group.}$$

$\forall\, v \in \mathrm{Val}(F)$, we have corresponding sets $\Psi(G_v)$, S_{ψ_v}, $\mathcal{S}_{\psi_v}$ together with a finite packet

$$\Pi_{\psi_v} \longrightarrow \Pi_{unit}(G_v)$$

of irred. unitary reps. of $G(F_v)$, with a canonical mapping

$$\pi_v \longrightarrow \langle \cdot, \pi_v \rangle$$

from Π_{ψ_v} to $\hat{\mathcal{S}}_{\psi_v}$ (linear chars. on $\mathcal{S}_{\psi_v}$).

Any $\psi \in \Psi(G)$ then has localizations $\psi_v \in \Psi(G_v)$ and homomorphisms $x \in \mathcal{S}_\psi \longmapsto x_v \in \mathcal{S}_{\psi_v}$, and hence

$$\Pi_\psi = \{\pi = \bigotimes_v \pi_v : \pi_v \in \Pi_{\psi_v},\ \langle \cdot, \pi_v \rangle = 1 \text{ for a.e. } v\}$$

with a linear character

$$\langle x, \pi \rangle = \prod_v \langle x_v, \pi_v \rangle \qquad x \in \mathcal{S}_\psi.$$

Note: Three simplifications:

1) Use L_F
2) $\Psi(G)$ is replaced by a quotient $\hat{\Psi}(G)$
3) $\Psi(G_v)$ should be $\Psi^+(G_v)$ to account for failure of Ramanujan conj. for $GL(n)$.

Global Theorem:

$$L^2_{disc}(G(F)\backslash G(\mathbb{A})) \simeq \bigoplus_{\psi \in \Psi_2(G)} \bigoplus_{\left\{\substack{\pi \in \Pi_\psi : \\ \langle \cdot, \pi \rangle = \varepsilon_\psi}\right\}} m_\psi \pi$$

where $\Psi_2(G) = \{\psi \in \Psi_2(G) : |S_\psi| < \infty\}$ and $m_\psi = \{1, 2\}$, and $\varepsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}$ a linear char. defined explicitly in terms of symplectic ε-factors.

Supplementary Global Theorem:

a) Suppose that ϕ lies in the subset

$$\Phi_{sim}(G) = \left\{ \psi \in \Psi(G) : \psi|_{SU(2)} = 1,\ S_\psi / Z(\hat{G})^\Gamma = 1 \right\}$$

Then $\hat{G}$ is orthogonal iff $L(s, \phi, Sym^2)$ has a pole at $s=1$.

Then $\hat{G}$ is symplectic iff $L(s, \phi, \Lambda^2)$ has a pole at $s=1$.

b) Suppose $\phi_i \in \Phi_{sim}(G_i)$, $i=1,2$. Then

$$\varepsilon(1/2, \phi_1 \times \phi_2) = 1$$

if $\hat{G}_1 \times \hat{G}_2$ are both orthog. or both symplectic.

2. Initial Applications:

F local: • Any $\pi \in \Pi_\psi$ is unitary
- local Langlands classification for G (almost)

F local or global: Whittaker models for local and global packets.

F global: • Rankin-Selberg L-functions for orthogonal or symplectic groups G: functoriality from G to GL(n)

- Analytic behavior of symmetric or skew-symmetric L-functions
- G has no embedded eigenvalues.
- Multiplicity 1: $\pi \in \Pi_\phi$ has mult. 1 except in case D_n
- Influence of symplectic root numbers on multiplicities.