Elliptic curves over function fields 3

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Overview

The goal today is to discuss the connection between Tate's conjectures and the BSD conjecture. It turns out that all of the objects entering into the BSD and Ш conjectures are "subquotients" of objects appearing in the Tate and Artin-Tate conjectures. Thus our results on the latter give results on BSD.

Elliptic curves and elliptic surfaces

Suppose $K = k(C)$ is a function field and E is an elliptic curve over K. Then there is an essentially unique surface $\mathcal{E}$ with a relatively minimal morphism $\pi : \mathcal{E} \to C$ whose generic fiber is E/K.

If E is constant, say $E \cong E_0 \times_k K$, then this is trivial: $\mathcal{E} = E_0 \times C$. To simplify, we will mostly ignore this case below.

To contruct $\mathcal{E}$ in the general case, choose any Weierstrass model for E:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad a_i \in K \qquad (*)$$

Let $U \subset C$ be a non-empty open subset such that the a_i are regular on U and such that the equation is minimal for all $x \in U$. Then (*) made homogeneous defines a hypersurface V in $\mathbb{P}^2 \times U$ with a morphism $V \to U$ whose fibers are projective plane cubics.

At each of the finitely many points of C not in U, we change Weierstrass equation to get a model which is defined and minimal over some other open U' containing the point in question. Working as above, we get an open surface V' mapping to U'. We can glue these over $U \cap U'$. (The change of coords we used gives the glueing recipe.)

Repeating for each missing point, we arrive at a surface $\mathcal{W}$ mapping surjectively to C whose generic fiber is E/K. ($\mathcal{W}$ is for Weierstrass.)

The surface $\mathcal{W}$ is not quite what we want because it might be singular. To fix this, we blow up singular points to arrive at a non-singular surface $\mathcal{E} \to C$ with generic fiber E/K. (The singularities and the configurations of curves that appear were classified by Néron and Kodaira.)

Example

Take $K = k(t)$ with $\mathrm{char}(k) > 3$ and E given by

$$y^2 + xy = x^3 - t^4$$

The discriminant of this model is $\Delta = t^4(1 - 16 \cdot 27 \cdot t^4)$ and so it is minimal at all $x \in U = \mathbb{A}^1 \subset C = \mathbb{P}^1$. So we consider

$$V = \{y^2z + xyz = x^3 - t^4z^3\} \subset \mathbb{P}^2 \times U.$$

To bring in the fiber over infinity, let $u = 1/t$ and change coords so that the Weierstrass equation is

$$y'^2 + ux'y' = x'^3 - u^2$$

with $\Delta = (u^4 - 16 \cdot 27)u^4$. This model is minimal at all finite values of u, so we set

$$V' = \{y'^2 z' + ux'y'z' = x'^3 - u^2 z'^3\} \subset \mathbb{P}^2 \times U'.$$

where $U' = \{u \neq \infty\} = \{t \neq 0\} \subset \mathbb{P}^1$

To get $\mathcal{W}$, we glue $V \setminus \{t = 0\}$ and $V' \setminus \{u = 0\}$ by identifying $([x', y', z'], u)$ with $([t^{-2}x, t^{-3}y, z], t^{-1})$. The result is $\mathcal{W} \to \mathbb{P}^1$.

To obtain $\mathcal{E}$, we have to resolve the singular points. It turns out that there are two, one each in the fibers over $t = 0$ and $t = \infty$. (The fiber over $t = 1/(16 \cdot 27)$ is singular, but the surface is non-singular there.) It's a fun exercise to do the explicit blow ups. (Hint: the reduction types are I_4 and IV.)

Points and divisors

Take E/K and $\mathcal{E} \to C$ as above. To each point of $E(K)$ we can associate a divisor on $\mathcal{E}$ (a section) by taking the Zariski closure. This turns out to give a well-defined map $E(K) \to NS(\mathcal{E})$, but this is not in general a group homomorphism! It turns out to be better to proceed in the other direction.

Before doing so, let's consider the irredubible curves on $\mathcal{E}$. Among them we have the smooth fibers, the irreducible components of the bad fibers. Any other irreducible curve on $\mathcal{E}$ will map finitely to $\mathcal{C}$ and so will be a section or a "multisection."

[picture]

NS and MW

Given a divisor D on $\mathcal{E}$, take its intersection with the generic fiber to get a divisor on E. The class of this divisor in $\mathrm{Pic}(E)$ only depends on the class of D in $NS(\mathcal{E})$.

We introduce a filtration on $NS(\mathcal{E})$ by letting $L^0 = NS(\mathcal{E})$, $L^1 =$ classes whose intersection with the generic fiber has degree 0, and $L^2 =$ the classes generated by the components of the fibers of π. It's obvious that $L^0/L^1 \cong \mathbb{Z}$ generated by the class of the zero section. It's not too hard to show that L^2 is the free group on the components of fibers not meeting the zero section, together with the class of any one fiber. The subquotient L^1/L^2 is the interesting part:

Shioda-Tate: (E non-constant) $L^1/L^2 \cong \mathrm{Pic}^0(E) = E(K)$.

The numerical version of this says

$$\mathrm{Rank}\, NS(\mathcal{E}) - \mathrm{Rank}\, E(K) = 2 + \sum_x (f_x - 1)$$

Cohomology

The interesting part of the cohomology of $\mathcal{E}$ is $H^2(\overline{\mathcal{E}}, \mathbb{Q}_\ell)$. This carries a filtration whose graded pieces are

$$H^2(\overline{C}, \mathbb{Q}_\ell) \qquad H^1(\overline{C}, R^1\pi_*\mathbb{Q}_\ell) \qquad H^0(\overline{C}, R^2\pi_*\mathbb{Q}_\ell)$$

The first and last of these are easily made explicit and are accounted for (wrt T_1) by the zero section and the components of the fibers. The middle group is interesting.

Full details are too complicate to summarize here, but it turns out that

$$H^1(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1))^G \cong \mathrm{Sel}_\ell(E) := \operatorname{proj}\lim_n Sel(E, [\ell^n])$$

and

$$\mathrm{Rank}_{\mathbb{Z}_\ell} H^2(\overline{\mathcal{E}}, \mathbb{Z}_\ell(1))^G - \mathrm{Rank}\,\mathrm{Sel}_\ell(E) = 2 + \sum_x (f_x - 1).$$

(The starting point is that the generic stalk of $R^1\pi_*\mathbb{Q}_\ell$ is the dual of the Tate module of E.)

Similarly, using the Leray spectral sequence to compute $Br(\mathcal{E}) = H^2(\mathcal{E}, \mathbb{G}_m)$, one finds that

$$Br(\mathcal{E}) \cong Ш(E)$$

Zetas

Finally, let's compare the zeta function of $\mathcal{E}$ with the L-function of E, first in an elementary way, then in a fancy way. To keep the notation under control, let's assume that all bad fibers of π are over k-rational points of C and that all components of the bad fibers are k-rational.

$$\begin{aligned} Z(\mathcal{E}, T) &= \prod_{x} (1 - T^{\deg(x)})^{-1} \\ &= \prod_{y} \prod_{\pi(x)=y} (1 - T^{\deg(x)})^{-1} \\ &= \prod_{y} Z(\pi^{-1}(y), T) \end{aligned}$$

where y runs through closed points of C.

Thus

$$Z(\mathcal{E},T)=\prod_{\text{good}y}\frac{1-a_yT+q_yT^2}{(1-T)(1-q_yT)}\prod_{\text{bad}y}\frac{1-a_yT}{(1-T)(1-q_yT)^{f_y}}$$

This is

$$\frac{Z(C,T)Z(C,qT)}{L(E,T)(1-qT)^{(\sum f_y-1)}}$$

Unwinding this using what we know about zetas (RH in particular), we find that

$$P_2(T)=L(E,T)(1-qT)^{2+\sum(f_y-1)}$$

and

$$-\operatorname{ord}_{s=1}\zeta(\mathcal{E},s)-\operatorname{ord}_{s=1}L(E,s)=2+\sum(f_y-1)$$

Zetas again

Let $\mathcal{F} = R^1\pi_*\mathbb{Q}_\ell$. Our definition of $L(E,T)$ is equivalent to

$$L(E,T) = \prod_y \det(1 - T\,\mathrm{Fr}_y\,|\mathcal{F}_y)^{-1}$$

Thus by the Grothendieck-Lefschetz trace formula, we have

$$L(E,T) = \prod_{i=0}^{2} \det(1 - T\,\mathrm{Fr}_q\,|H^i(\overline{C},\mathcal{F}))^{(-1)^{i+1}}$$

When E is non-constant, the H^0 and H^2 vanish and we have

$$L(E,T) = \det(1 - T\,\mathrm{Fr}\,|H^1(\overline{C},\mathcal{F}))$$

This shows (again) that $\mathrm{ord}_{s=1} L(E,s) \geq \mathrm{Rank}_{\mathbb{Z}_\ell} \mathrm{Sel}_\ell(E)$.

Summary

Combining today's results with what we know about the Tate conjecture, we have:

- Rank $E(K) \leq \text{Rank}_{\mathbb{Z}_\ell} \text{Sel}_\ell(E) \leq \text{ord}_{s=1} L(E,s)$
- Equality holds iff $|\text{Ш}(E)| < \infty$.
- Equality holds iff the Tate conjecture holds for the associated surface $\mathcal{E}$.
- In particular, equality holds when $\mathcal{E}$ is dominated by a product of curves (e.g., products of curves, rational surfaces) or is a K3 surface.

[Rmk: Everything today has an analog for general curves over function fields and their Jacobians.]

A useful class of examples

Let X be a smooth curve over $K = k(t)$. Suppose there exists $g \in k[t,x,y] \subset K[x,y]$ which is the sum of exactly 4 non-zero monomials satisfying mild conditions on the exponents (see below) such that $K(X) \cong \mathrm{Frac}(K[x,y]/(g)$. Then the BSD conjecture holds for the Jacobian of X.

There are interesting such curves of every genus. E.g., if $\mathrm{char}(k) \not| (2g+1)(2g+2)$, consider the hyperelliptic curve

$$y^2 = x^{2g+2} + x^{2g+1} + t^d$$

of genus g. If $d \neq 0$ in k, this satisfies the hypotheses.

Write the exponents of g in a 4×3 matrix and add a 4th column so that the row sums are 1. Call this guy A. Assume that $\det(A) \neq 0$. Let B be the integer matrix such that $AB = \delta$ with δ minimal positive intger. Assume that $\delta \neq 0$ in k.

Under these hypotheses, after possibly extending k, there are dominant rational maps

$$F_\delta^2 \to \mathcal{X} \to F_1^2$$

where F_δ^d denotes the Fermat variety of dimension d and degree δ and $\mathcal{X}$ is the surface associated to X.

There also dominant rational maps

$$F_\delta^1 \times F_\delta^1 \to F_\delta^2.$$

Thus $\mathcal{X}$ is dominated by a product of curves, it satisfies the Tate conjecture, and the Jacobian of X satisfies the BSD conjecture.

Next time we'll show that there are large analytic ranks in the example above as d varies and so conclude that there are large algebraic ranks too.