2. Stark's basic conjecture.

2.1 The S-unit theorem. In this section we deal mainly with one number field. We denote it by K and its places by w because the concepts we discuss will be applied to the top field K of a Galois extension K/k. Let S be a finite set of places of K containing the set S_∞ of infinite places. It is often convenient to work with the Dedekind ring of "S-integers"

$$\mathcal{O}_S = \{\alpha \in K \mid |a|_w \leq 1, \quad \text{for all } \ w \notin S\},$$

the localization of $\mathcal{O}_K$ whose primes "are" those of $\mathcal{O}_K$ not in S. The group $\mathcal{O}_S^*$ of "S-units", the elements $u \in K$ such that $|u|_w = 1$ for all $w \notin S$, plays a key role in Stark's conjectures. In this section we recall the S-unit theorem and establish some notation.

Let $Y = Y_S$ denote the free $\mathbb{Z}$-module with basis the places in S. For $y = \sum_{w \in S} n_w w \in S$, put $\epsilon(y) = \sum_{w\in S} n_w$ and let $X = X_S = \mathrm{Ker}(\epsilon)$, so that we have an exact sequence

$$\{0\} \to X \to Y \to \mathbb{Z} \to \{0\} \tag{2.1.1}$$

The S-unit theorem concerns the map $\lambda = \lambda_S$

$$\lambda : \mathcal{O}_S^* \to \mathbb{R}Y_S \tag{2.1.2}$$

defined by $\lambda(u) = \sum_{w\in S} \log |u|_w \cdot w$.

Notation: $\mathbb{R}Y$ is the real vector space with basis $\{w\}, w \in S$. More generally, for a ring R and an abelian group A we will denote the R-module $R \otimes_{\mathbb{Z}} A$ simply by RA. Also, by $|\ |_w$ we denote the the normed absolute value at the place w. We define $|\alpha|_w$ as the factor by which the additive Haar measure in the completion K_w is stretched under multiplication by α. For complex w this is the square of the usual absolute value value, and does not satisfy the triangle inequality.

For $\alpha \neq 0$ the product formula $\prod_{\text{all } w} |\alpha|_w = 1$ holds and shows that $\lambda(u) \in \mathbb{R}X_S$ for all S-units u,

(2.1.3) Theorem. The kernel of λ is the finite cyclic group $\mu(K)$ of roots of 1 in K. The image of λ is a lattice spanning $\mathbb{R}X$.

Let $r = r_S = \dim \mathbb{R}X = |S| - 1$, and let $\{u_1, u_2, ..., u_r\}$ be a fundamental system of S-units, that is, units whose images $\lambda(U_i)$ form a basis for the lattice $\lambda(\mathcal{O}_S^*)$. Let $S = \{w_0, w_1, ..., w_r\}$. The *S-regulator*

$$R_S := \text{ absolute value of } \ \det\nolimits_{1\leq i,j\leq r} \ \log |u_i|_{w_j} \tag{2.1.4}$$

is the covolume of the lattice in $\mathbb{R}X$ in the measure $|dx_1 dx_2 \cdots dx_r|$ on X, which by symmetry is independent of our numbering of the w_j

2.2 Incomplete L-functions. Now we assume the same setup as earlier: K/k finite Galois with group G; V a representation of G; v a variable place of k and w a place of K above v; G_w and I_w are the decomposition and inertia groups for w over k. Let S be a finite set of places of k containing the set S_∞ of infinite places and for an extension field k' of k, let $S_{k'}$ denote the set of plces v' of k' above S.

It is important to consider the Euler product

$$L_S(s, V) = \prod_{v \notin S} L_v(s, V) \tag{2.2.1}$$

whose factors are those of $L(s,V)$ corresponding to prime ideals of $\mathcal{O}_k$ not in S. The "induction rule"

$$L_S(s, \mathrm{Ind}_{k'}^{k} W) = L_{S_{k'}}(s, W), \tag{2.2.2}$$

still holds for these incomplete L_S's, because we proved (1.9.2) for each individual Euler factor L_v, not only for their product.

Stark's conjectures concern the leading term of the Taylor expansion of $L_S(s,V)$ at $s = 0$, which we denote by $c_S(V)s^{r_S(V)}$. Thus, $r_S(V)$ is the order of vanishing of $L_S(s,V)$ at $s = 0$, and

$$c_S(V) = \lim_{s\to 0} \frac{L_S(s,V)}{s^{r_S(V)}} \neq 0. \tag{2.2.3}$$

2.3 Order of zero. In this paragraph we prove

(2.3.1) Theorem. $r_S(V) = -\dim V^G + \sum_{v\in S} \dim V^{G_w}$.

Proof. We get this from the functional equation and the very simple behavior near $s = 1$. At $s = 1$ none of the local L functions $L_v(s,V)$ has a zero or pole, as is obvious from their definition. Hence, $\Lambda(s,V)$, $L(s,V)$, and $L_S(s,V)$ for all S have the same order at $s = 1$. This order is $-\dim V^G$, or, equivalently, $\zeta_k(s)$ has order -1 (simple pole), and for each other irreducible representation V of G, $L(1,V) \neq 0, \infty$. For an abelian character ψ this is an old story, starting with Dedekind's proof of his theorem on primes in arithmetic progressions, and the non-abelian case follows from that via (1.7.3) and Exercise 1.7.5.

Now for s near 0, the functional equation (1.8.5) for $\Lambda(s,V) = L_S(s,V)\prod_{v\in S} L_v(s,V)$, namely

$$\Lambda(1-s,V) = CB^s\Lambda(s,V*),$$

shows that our theorem follows immediately from the fact that for every place v of k the local function $L_v(s,V)$ has a pole of order $\dim V^{G_w}$. This is easily checked from the definitions, using for infinite v the fact that $s\Gamma(s) = \Gamma(s+1) > 0$ for $s > -1$, and for finite v that the order of the pole is the multiplicity of 1 as eigenvalue of the action of σ_w on V^{I_w}, and $V^{G_w} = (V^{I_w})^{<\sigma_w>}$.

2.4 The zeta function. The incomplete zeta function

$$\zeta_{k,S}(s) = \prod_{v\notin S}(1 - q_v^{-s})^{-1},$$

has a zero of order $r_S = |S| - 1$ at $s = 0$, because each 'dim' in (2.3.1) is 1. The leading coefficient is also an old story.

(2.3.1) Theorem. As $s \to 0$, $\zeta_S(s) \sim \frac{h_S R_S}{w} s^{|S|-1}$.