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Stark's Basic Conjecture

John Tate

Some Notation.

In these notes, k denotes a number field, i.e., a finite extension of the field $\mathbb{Q}$ of rationals, and O_k is its ring of integers. If I is an ideal in O_k, we denote its number of residue classes $|O_k/I|$ by $\mathbb{N}I$. This number is also the positive generator of the ideal $N_{K/\mathbb{Q}}I \subset \mathbb{Z}$, and is a multiplicative function of ideals. By a place v of k we mean an equivalence class of non-trivial absolute values on k. These are of three types, finite, real and complex, corresponding, respectively, to a prime ideal P in O_k, to an embedding of k into $\mathbb{R}$, and to a pair of distinct complex conjugate embeddings of k into $\mathbb{C}$. We denote the number of real places by $r_1 = \mathrm{r}_1(k)$ and the number of complex ones by $r_2 = r_2(k)$. For each finite place v we denote the corresponding prime ideal by P_v and the residue field by $\mathbb{F}_v$. We sometimes denote the cardinality $|\mathbb{F}_v|$ of $\mathbb{F}_v$ by $q_v = \mathbb{N}P_v$.

1. L-functions.

The L-functions associated with number fields are of two fundamentally different types. We will first discuss the very classical abelian ones, made with characters of generalized ideal class groups, which go back to Dirichlet and are the simplest sort of "automorphic" L-functions. Then we will discuss the "nonabelian" L-functions introduced by E. Artin, which are made with representations of Galois groups, and were the first examples of explicitly defined "motivic" L-functions. Artin L-functions made with one dimensional representations of a Galois group are equal to classical abelian L-functions, thanks to Artin's reciprocity law.

1.1. Ideal class characters.

Define a "level" to be a pair $\mathfrak{m} = (\mathfrak{m}_f, \mathfrak{m}_\infty)$ consisting of an ideal $\mathfrak{m}_f$ in O_k and a set $\mathfrak{m}_\infty$ of real places of k. For $a \in O_k$, write $a \equiv b \bmod^\times \mathfrak{m}$ if a and b are prime to $\mathfrak{m}$, $a \equiv b \bmod \mathfrak{m}_f$ and $a/b > 0$ at each place in $\mathfrak{m}_\infty$. A (generalized) ideal class character mod $\mathfrak{m}$ is a multiplicative function χ of ideals I prime to $\mathfrak{m}$ such that $\chi(I) = 1$ if I is a principal ideal generated by a number $a \equiv 1 \bmod^\times \mathfrak{m}$. For example, if $k = \mathbb{Q}$, then a character mod $(m\mathbb{Z}, \{\infty\})$ is the same as a Dirichlet character mod m, if we identify ideals prime to m with their positive integer generators.

Exercise: (a) Let $\mathcal{I}_\mathfrak{m}$ denote the group of fractional ideals prime to $\mathfrak{m}$, i.e., generated by prime ideals not dividing $\mathfrak{m}_f$. Let $\mathcal{P}_\mathfrak{m}$ denote the subgroup of the principal ideals which are generated by elements $c \in k^*$ of the form $c = \frac{a}{b}$ with $a \equiv b \bmod^\times \mathfrak{m}$. Show that a character mod $\mathfrak{m}$ in the above sense is the same as the restriction to

Department of Mathematics, The University of Texas at Austin, 1 University Station – C1200, Austin, TX 78712-0257
E-mail address: tate@math.utexas.edu

integral ideals of a homomorphism $\chi : \mathcal{I}_\mathfrak{m} \to \mathbb{C}^*$ which is trivial on $\mathcal{P}_\mathfrak{m}$, hence is the same as a character of the group $\mathcal{C}_\mathfrak{m} := \mathcal{I}_\mathfrak{m}/\mathcal{P}_\mathfrak{m}$.

(b) Show that there is an exact sequence

$$\mathcal{O}_k^* \to (\mathcal{O}_k/\mathfrak{m}_f)^* \times \prod_{v \in \mathfrak{m}_\infty} k_v^*/(k_v^*)_{>0} \to \mathcal{C}_\mathfrak{m} \to \mathcal{C} \to 0$$

where $\mathcal{C} = \mathcal{C}_{(\mathcal{O}_k, \emptyset)}$ is the usual ideal class group. It follows that the groups $\mathcal{C}_\mathfrak{m}$ are finite. Hence the values of an ideal class character χ are complex roots of 1, and in particular, have absolute value 1.

(c) The set of levels is partially ordered under the relation

$$\mathfrak{m}_1 \leq \mathfrak{m}_2 \;\Leftrightarrow\; (\mathfrak{m}_1)_f | (\mathfrak{m}_2)_f \text{ and } (\mathfrak{m}_1)_\infty \subset (\mathfrak{m}_2)_\infty$$

If $\mathfrak{m}_1 \leq \mathfrak{m}_2$, the identity map on ideals induces a homomorphism $\mathcal{C}_{\mathfrak{m}_2} \to \mathcal{C}_{\mathfrak{m}_1}$. Show this homomorphism is surjective (even if there are primes dividing $\mathfrak{m}_2$ which do not divide $\mathfrak{m}_1$). This allows us to identify characters mod $\mathfrak{m}_1$ with certain characters mod $\mathfrak{m}_2$ and to view the group of all ideal class characters as the Pontrjagin dual of the profinite group $\operatorname{proj\,lim}_\mathfrak{m} \mathcal{C}_\mathfrak{m}$. Show for each character χ that there is a smallest level $\mathfrak{m}$ such that χ is a character mod $\mathfrak{m}$. This level is called the *conductor* of χ and denoted by $\mathfrak{m}_c$. One says χ is *primitive* mod $\mathfrak{m}_\chi$, and *imprimitive* mod $\mathfrak{m}$ for all $\mathfrak{m} > \mathfrak{m}_\chi$.

1.2. Classical abelian L-functions.

Associated with each ideal class character χ is a function $L(s,\chi)$ of a complex variable s, defined in the right half-plane $\Re(s) > 1$ by

$$L(s,\chi) = \prod_P \frac{1}{1-\chi(P)\mathbb{N}P^{-s}} = \sum_I c(I)\mathbb{N}I^{-s}, \tag{1.2.1}$$

where the product is over the prime ideals P not dividing $\mathfrak{m}_\chi$ and the sum over all integral ideals I prime to $\mathfrak{m}_\chi$. The formal equality of the product and sum follows from unique factorization and the fact that the function of ideals $X(I) := \chi(I)\mathbb{N}I^{-s}$ is multiplicative , so that for $x > 0$

$$\prod_{\mathbb{N}P \leq x} (1 - X(P))^{-1} = \prod_{\mathbb{N}P \leq x} \sum_{n_P = 0}^{\infty} X(P)^{n_P}$$
$$= \sum_{(\dots, n_P, \dots)} \prod_{\mathbb{N}P \leq x} X(P)^{n_P} = \sum_{(\dots, n_P, \dots)} X(\prod_{\mathbb{N}P \leq x} P^{n_P}) = \sum_I X(I),$$

where the last sum is over all integral ideals I prime to $\mathfrak{m}_\chi$ whose prime factorization involves only prime ideals with norm $\leq x$. The absolute and uniform convergence in $\Re(s) \geq \sigma > 1$ as $x \to \infty$ follows from

$$\sum_P |X(P)| = \sum_P \mathbb{N}P^{-\sigma} \leq [k:\mathbb{Q}] \sum_p p^{-\sigma} \leq \sum_{n=1}^{\infty} n^{-\sigma} \leq \int_1^\infty x^{-\sigma} dx = \frac{1}{\sigma - 1}.$$

Hecke proved that $L(s,\chi)$ has an analytic continuation to the whole s-plane if $\chi \neq 1$. For $\chi = 1$, the zeta function $\zeta_k(s) := L(s,1)$ is analytic in the whole plane except for a simple pole at $s = 1$. The proof of analytic continuation gives also a functional equation relating $L(1-s,\chi)$ and $Ls,\bar{\chi})$. One way to express this functional equation is the following. For each infinite place v of k, define $\gamma_v(s,\chi)$ to be $\Gamma(\frac{s}{2})$ if v is real and not contained in the conductor of χ, to be $\Gamma(\frac{s+1}{2})$ if v

is real and in the conductor of χ, and $\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\pi^{1/2}\Gamma(s)$ if v is complex. Then there exist canstants $B_\chi > 0$ and $C_\chi \in \mathbb{C}^*$ such that the functions

$$\Lambda(s,\chi) := \prod_{v\,|\,\infty} \gamma_v(s,\chi)L(s,\chi) \tag{1.2.2}$$

satisfy

$$\Lambda(1-s,\chi) = C_\chi B_\chi^s \Lambda(s,\bar{\chi}). \tag{1.2.3}$$

1.3. Representations of finite groups.

Let G be a finite group. By a *representation* of G we mean a finite dimensional left $\mathbb{C}[G]$-module V. To give such representation is the same as to give a finite dimensional $\mathbb{C}$-vectorspace V, together with a homomorphism $\rho : G \to GL(V)$ giving the action of G on V. (We use the symbol $GL(V)$ to denote $\mathrm{Aut}(V)$ because a basis for V gives an isomorphism $\mathrm{Aut}(V) \xrightarrow{\sim} GL_n(\mathbb{C})$ for $n = \dim(V)$.) But we take the module point of view in which the symbol V includes the action of G, and write simply σx instead of $\rho(\sigma)x$ for $x \in V$ and $\sigma \in G$. Two representations V and W are said to be isomorphic if they are isomorphic as $\mathbb{C}[G]$-modules. We assume the reader knows the basic theory and simply list the results we need and some notation and terminology we will use.

(i) Examples. a) The *regular* representation is $V = \mathbb{C}[G]$ with the action of G given by left multiplication.

b) $V = \mathbb{C}$, with $\sigma x = x$ for all $\sigma \in G, x \in \mathbb{C}$ is the *trivial* representation.

c) If V and V' are representations, then $V \otimes_{\mathbb{C}} V'$ is also, with $\sigma(x \otimes x') = \sigma x \otimes \sigma x'$.

d) The space $\mathrm{Hom}_{\mathbb{C}}(V, V')$ of all linear maps $f : V \to V'$ is a representation, with $(\sigma f)(x) = \sigma(f(\sigma^{-1}x))$. (Note: We often write f^σ instead of σf, so $f^\sigma(x) = \sigma(f(\sigma^{-1}x))$. But remember: $(f^\sigma)^\tau = f^{(\tau\sigma)}$. Our action is always a left action.)

e) The *dual* representation to V is $V^* := \mathrm{Hom}(V, \mathbb{C})$. We have $(V^*)^* = V$, and $\mathrm{Hom}_{\mathbb{C}}(V^*, V') \xrightarrow{\sim} \mathrm{Hom}((V')^*, V)$, canonically. Also, $\mathrm{Hom}(V, V') \xrightarrow{\sim} V^* \otimes_{\mathbb{C}} V'$, canonically.

(ii) Semisimplicity. $\mathbb{C}[G]$ is semisimple, i.e., every representation is a direct sum of irreducible representations. (V is *irreducible* if it is a simple $\mathbb{C}[G]$-module, i.e., has exactly two submodules, V and $\{0\}$.)

(iii) Characters. The *character* of a representation V is the function $\chi_V : G \to \mathcal{C}$ defined by $\chi_V(\sigma) = \mathrm{Tr}(\sigma \mid V)$, the trace of the map $x \mapsto \sigma x$ of V into V. This χ_V is a *central function* on G, meaning that it is constant on each conjugacy class, or, equivalently, that the element $\sum_{\sigma\in G} \chi(\sigma)\sigma$ is in the center of $\mathbb{C}[G]$. The *degree* of the character $\chi$of a representation V is $\chi(1) = \dim V$. Examples.

$$\chi_{\mathbb{C}}(\sigma) = 1, \quad \text{for all } \sigma \in G. \tag{a}$$

$$\chi_{\mathbb{C}[G]}(1) = |G|, \quad \text{and} \quad \chi_{\mathbb{C}[G]}(\sigma) = 0 \quad \text{for} \quad \sigma \neq 1. \tag{b}$$

$$\chi_{V\otimes_{\mathbb{C}}V'}(\sigma) = \chi_V(\sigma) \cdot \chi_{V'}(\sigma). \tag{c}$$

$$\chi_{V^*}(\sigma) = \chi_V(\sigma^{-1}) = \bar{\chi}_V(\sigma). \tag{d}$$

($\chi_V(\sigma)$ is a sum of complex m-th roots of 1, if $\sigma^m = 1$.)

$$\chi_{\mathrm{Hom}(V,V')} = \chi_{V^*\otimes V'} = \bar{\chi}_V(\sigma)\chi'_V(\sigma). \tag{e}$$

(iv) Orthogonality. One makes the space C of central functions on G into a Hilbert space by putting $\langle f, g\rangle_G = \frac{1}{|G|}\sum_{\sigma\in G} f(\sigma)\bar{g}(s)$.

$$\langle \chi_V, \chi_{V'}\rangle_G = \dim(\mathrm{Hom}_G(V, V'), \tag{1.3.1}$$

Let $\{V_i\}$ be a complete set of non-isomorphic representations of G, and let χ_i be the character of V_i. These χ_i form an orthonormal basis for C. Thus the number of isomorphism classes of irreducible representations V_i is equal to $\dim C$, that is, to the number of conjugacy classes in G.

(v) Isotypical components; projections. Let V be a representation. Then by semisimplicity,

$$V = \oplus_i V[i], \tag{1.3.2}$$

where $V[i]$ is isomorphic to a direct sum of copies of V_i. The decomposition (1.3.2)) is unique. If $x = \sum_i x_i$ with $x_i \in V[i]$, then

$$x_i = \frac{\dim V_i}{|G|}\sum_{\sigma\in G} \bar{\chi}_i(\sigma)\sigma x, \tag{1.3.3}$$

In other words, $\frac{\dim V_i}{|G|}\sum_{\sigma\in G}\bar{\chi}_i(\sigma)\sigma \in \mathbb{C}[G]$ is the projection of V onto $V[i]$ which kills all the other $V[j]$'s.

The $V[i]$'s are called the *isotypical* components of V. The decomposition of $V[i]$ into a direct sum of copies of V_i corresponds to expressing $\mathrm{Hom}_G(V_i, V)$ into a direct sum of 1-dimensional subspaces and is not at all unique, but the number of copies of V_i in such an expression, the *multiplicity* of V_i in V, is unique and equals $\langle\chi_i, \chi_V\rangle_G. = \dim \mathrm{Hom}_G(V_i, V)$.

One denotes the isotypical component of V corresponding to the trivial representation $\mathbb{C}$ by $V^G = \{x \in V \mid \sigma x = x$ for all $\sigma \in G$. the corresponding projection, $\frac{1}{|G|}\sum_\sigma \sigma$ is especialy useful.

(v1) Induced representation; Frobenius reciprocity. Let Rep_G denote the category of representations of G. Let $f : H \to G$ be a homomorphism of groups. In this paragraph we discuss adjoint functors $f^* : \mathrm{Rep}_G \to \mathrm{Rep}_H$ and $f_* : \mathrm{Rep}_H \to \mathrm{Rep}_G$. The classic case is that in which f is an inclusion $H \subset G$. In that case $f * V$ is called the restriction of V to H and denoted by $\mathrm{res}\, V$ or $\mathrm{res}_H^G V$, and f_*W is called the representation of G *induced by* the representation W of the subgroup H, and is denoted by $\mathrm{Ind}\, W$ or $\mathrm{Ind}_H^G W$. In general, if V is a representation of G, then f^*V is the representation of H with the same underlying vector space as V, with H acting through f, that is, $\tau x = f(\tau)x$ for $\tau \in H$ and $x \in V$. The functor f_* in the other direction comes with canonical elements $\iota_W \in \mathrm{Hom}_H(W, f^*f_*W)$ and is characterized by the fact that for every pair of representations W of H and V of G, the map $\mathrm{Hom}_G(f_*W, V) \to \mathrm{Hom}_H(W, f^*V)$ given by $\phi \mapsto (f^*\phi)\circ\iota_W$ is bijective. Since every group homomorphism f is a passage to quotiont followed by an isomorphism followed by an inclusion, it suffices. in order to describe f_* explicitly, to treat the two cases $H \to H/N$ and $H \subset G$. In the first of these, $f_*W = W^N$, and ι_W is the projection of W onto W^N j discussed in (v) just above. In the second, classical, case, ι_W is an inclusion $W \subset f_*W$, and $\mathrm{Ind}_H^G = f_*W = \oplus_j\sigma_j W$, where the σ_j are representatives of the left cosets $\sigma_j H$ of H in G. The action of G is forced by this: $\sigma\sum_i \sigma_i y_i = \sum_i \sigma_{\sigma(i)}\tau_i y_i$ for $y_i \in W$, where the subscript $\sigma(i)$ and $\tau_i \in H$ are uniquely determined by $\sigma\sigma_i = \sigma_{\sigma(i)}\tau_i$.

If χ is the character of V, we denote by $f^*\chi$ the character of f^*V. Clearly, $f^*\chi(\sigma) = \chi(f(\sigma))$. If *psi* is the character of W, we denote by $f_*\psi$ the character of f_*W. It is not hard to see that

$$(f_*\psi)(\sigma) = \frac{1}{|H|}\sum_{\rho\in G}\sum_{\tau\in H; f(\tau)=\rho\sigma\rho^{-1}} \psi(\tau). \tag{1.3.4}$$

Equating the dimensions of the spaces on each side of the isomorphism $\mathrm{Hom}_G(f_*W, V) \xrightarrow{\sim} \mathrm{Hom}_H(W, f^*V)$ characterizing f_* gives the relation

$$\langle f_*\psi, \chi\rangle_G = \langle \psi, f^*\chi\rangle_H. \tag{1.3.5}$$

This is known as *Frobenius reciprocity*.

(vii) Abelian characters; duality for finite abelian groups. The characters χ of 1-dimensional representations are the same thing as the group homomorphisms $G \to \mathbb{C}^*$. We will call them *abelian characters*. They form an abelian group $\mathrm{Hom}(G, \mathbb{C})$ under multiplication. If G is abelian, they are the only irreducible characters. The group they form is called the *character group* of G and is denoted in these notes by $\hat{G}$. We have a perfect duality: the natural maps $G \to \hat{G}\hat{}$, and $\mathrm{Hom}(\hat{G}', \hat{G}) \to \mathrm{Hom}(G, G')$ are isomorphisms.

1.4. Decomposition, inertia, Frobenius.

Let K/k be a finite Galois extension and $G = G_{K/k}$ its Galois group. Recall that for each place v of k, G acts transitively on the set of places w of K which lie above (i.e., extend) v. The stabilizer of one of these w's is a subgroup of G called the *decomposition group* of w and denoted by G_w. It can be identified with the Galois group of the corresponding extension of the completions K_w/k_v.

For finite v, an element σ of G_w induces an automorphism $\tilde{\sigma}$ of the residue field $\mathbb{F}_w$ of w. The map $\sigma \mapsto \tilde{\sigma}$ is a homomorphism of G_w onto the Galois group of the w/v residue field extension. The kernel is called the *inertia group* of w and is denoted by I_w. The order of I_w is the ramification index $e_{w/v}$ and is 1 for all but a finite number of places v, those dividing the relative discriminant ideal $d_{K/k}$. As Galois group of the finite field extension $\mathbb{F}_w/\mathbb{F}_v$, the quotient group G_w/I_w is cyclic with a canonical generator, the Frobenius automorphism $\tilde{\sigma}_w$ which raises elements of $\mathbb{F}_w$ to the q_v-th power, where $q_v = \mathbb{N}P_v$ is the number of elements in $\mathbb{F}_v$.

1.5. Artin L-functions.

Let K/k be a finite Galois extension and $G = G_{K/k}$ its group. Let V be a representation of G. For a finite place v of k let $F_{v,V}(T) = \det(1 - \sigma_w T \mid V^{I_w})$ be the characteristic polynomial of the action on V^{I_w} of a Frobenius automorphism σ_w attached to a place w of K above v. Although σ_w is determined only up to multiplication by an element of I_w, its action on V^{I_w} is independent of which we chose. The polynomial $F_v(T, V)$ obviously depends only on the isomorphism class of V, and it also depends only on v, not on the choice of w above v, because the places w above v are all conjugate. Note that for v unramified in K, hence for all but a finite number of v we have $I_w = \{1\}$, hence $V^{I_w} = V$, and $F_v(T, V)$ is of degree $\dim V$. Note also that $F_v(T, V)$ depends only on V as G_w-module, not as G-module. Artin defined his L-function as an "Euler product" over the finite places v of k "local L-functions" $L_v(s,V) := F_v(\mathbb{N}P_v^{-s},V)^{-1}$.

$$L(s,V) = \prod_v L_v(s,V) = \prod_v \frac{1}{\det(1-\sigma_w q_v^{-s} \mid V^{I_w})}. \tag{1.5.1}$$

Exercise: Show that this product converges absolutely and defines an analytic function in the half plane $\Re)s) > 1$.

In fact, it has a meromorphic continuation to the whole plane, and a deep conjecture of Artin is that it is an entire function if the trivial representation does not occur in V — more about this soon. It is really enough to consider irreducible representations, because

$$L(s, V\oplus W) = L(s,V)L(s,W) \tag{1.5.2}$$

Exercise: Since $L(s,V)$ depends only on the isomorphism class of the representation V, we can also denote it by $L(s,\chi_V$, where $\chi_V : G \to \mathbb{C}$ is the character of V. Show for each finite place v that for $\chi = \chi_V$,

$$\log L_v(s,\chi) = \sum_{n=1}^{\infty} \frac{\mathrm{Tr}(\sigma_w^n \mid V^{I_w})}{nq_v^{ns}} = \sum_{n=1}^{\infty} \frac{\chi(\sigma_w^n)}{nq_v^n s},$$

where $\chi(\sigma_w^n) = \frac{1}{|I_w|}\sum_{\tau\in\sigma_w^n}$ is the average value of χ on the coset σ_w^n of I_w.

Another property of these L-functions concerns the situation in which K is contained in a larger Galois extension K' of k, so $k \subset K \subset K'$. Then $G = G_{K/k}$ is a quotient group of $G' := G_{K'/k}$. Let V' denote the G'-module with the same underlying vector space as V, with G' acting through G. Then

$$L(s,V') = L(s,V) \tag{1.5.3}$$

This is true because if we let w' be a place of K' above w, then $I_w = I'_{w'}G_{K'/K}$, hence $(V')^{I_{w'}} = V^I$, and $\sigma_w = \sigma_{w'}G_{K'/K}$. Property (1.5.3) shows that $L(s,V)$ really depends only on V viewed as module $\bar{V}$ for "the" absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ of k. Note that the isomorphism class of $\bar{V}$ as G_k-module is independent of how we view K/k as subextension of $\bar{k}/k$. The representations of the form $\bar{V}$ for some K/k and V are, up to isomorphism, simply the $\mathbb{C}[G_k]$–modules X of finite dimension over $\mathbb{C}$ for which the action map $G_k \times X \to X$ is continuous for the profinite (Krull) topology in G_k and the discrete topology in X (or the usual complex vector space topology in X – it's the same, because $GL_n(\mathbb{C})$ has "no small subgroups").

The really key property of Artin's $\mathcal{L}$-functions relates L-functions of different fields. Suppose k' is an intermediate field, $k \subset k' \subset K$, the fixed field of a subgroup H of G. Let W be a representation of H and Let $V = \mathrm{Ind}_H^G W$ be the representation of G induced by W. Then

$$L(s,W) = L(s,V) \qquad (V = \mathrm{Ind}_{k'}^k W). \tag{1.5.4}$$

Example: If $k' = K$, and $W = \mathbb{C}$ is the trivial representation of the identity subgroup of G, then $V = \mathbb{C}[G]$ is the regular representation of of G, and since $\mathbb{C}[G] \xrightarrow{\sim} \oplus_i V_i^{\dim V_i}$, where the V_i are (representatives of the isomorphism classes of) the irreducible representations of G, we have, by (1.5.2) and (1.5.4)

$$\zeta_K(s) = \prod_i L(s,V_i)^{\dim V_i} = \zeta_k(s)\prod_{i\neq 1}(L(s,V_i)^{\dim V_i}, \tag{1.5.5}$$

if we number the V_i so that V_1 is the trivial representation of G, for by (1.5.3) we have then $L(s, V_1) = \zeta_k(s)$. More generally, if k' is any intermediate field, and W the trivial representation of H, then $V = \mathbb{C}[G/H]$ is the permutation representation of G acting on the set G/H of cosets of H, and by Frobenius reciprocity, $V \xrightarrow{\sim} \oplus_i V_i^{m_i}$, where $m_i = \dim\ V_i^H$. In particular, $m_1 = 1$ and we have

$$\zeta_{k'}(s) = \zeta_k(s) \cdot \prod_{i \neq 1} L(s, V_i)_i^m, \tag{1.5.6}$$

Thus, if the $L(s, V_i)$ are entire funtions for $i \neq 1$ as Artin conjectured, then the zeta function of k divides the zeta function of every extension field k'/k, and the Riemann hypothesis for all zeta functions would imply it for all L-functions. It was Artin's investigation of the interrelations among the zeta functions of different fields, in particular the intermediate fields of a non-abelian Galois extension K/k which led him to define his new kind of L-functions.

Exercise: Fill in the details of the following proof of (1.5.4). Let v be a finite place of k and w be a place of K above v. Let $G = \coprod_{i=1}^r G_w \rho_i H$ be the expression of G as disjoint union of double cosets of G_w and H. Let $w_i = \rho_i^{-1} w$ and let v_i' be the place of k' below w_i. Then $v_1', v_2', ..., v_r'$ are the places of k' above v, so it suffices to show that

$$L_v(s, V) = \prod_{i=1}^r L_{v_i'}(s, W). \tag{1.5.7}$$

For each i, let $G_w = \coprod_{j=1}^{m_i} \tau_{ij}(G_w \cap H^{\rho_i})$. (Notation: For $\rho \in G$ and $X \subset G$ we write $X^\rho := \rho X \rho^{-1}$.) Then

$$G = \coprod_{i=1}^r \coprod_{j=1}^{m_i} \tau_{ij} \rho_i H.$$

Hence, by definition of induced representation, V contains W as an H-submodule isomorphic to, and

$$V = \oplus_{i=1}^r\ \oplus_{j=1}^{m_i}\ \tau_{ij} \rho_i W = \oplus_{i=1}^r V_i, \tag{1.5.8}$$

where $V_i = \oplus_{j=1}^{m_i} \tau_{ij} \rho_i W$ is a G_w-module for each i, and is in fact isomorphic to the G_w-module induced from the $G_w \cap H^{\rho_i}$-module $\rho_i W$. Applying the automorphism ρ_i to our situation, ($H \mapsto H^{\rho_i}$, $k' \mapsto \rho_i k'$, $W \mapsto \rho_i W$, etc.), we have by transport of structure, $L_{v'}(s, \rho_i W) = L_{v_i'}(s, \rho_i W)$, where v' is the place of $\rho_i k$ below w. Comparing with (1.5.7) and (1.5.8) and using (1.5.2) shows that we are now reduced to the local case, in which $G = G_w$.

Next we reduce to the local unramified case by showing that $V^I \xrightarrow{\sim} \mathrm{Ind}_{H/J}^{G/I}(W^J)$, where $I = I_w$,and $J = H \cap I$ are the inertia subgroups of G and H, and where we view H/I as subgroup of G/J by identifying H/J with HI/I in the obvious way. One way to do this is to factor the homomorphism $f : H \to G/I$ in two ways

$$H \hookrightarrow G \to G/I \text{ and } H \to H/J \hookrightarrow G/I.$$

Calculating $f_* W$ in two ways accordingly we find

$$f_* W \xrightarrow{\sim} (\mathrm{Ind}_H^G W)^I \xrightarrow{\sim} V^I \text{ and } f_* W \xrightarrow{\sim} \mathrm{Ind}_{H/J}^{G/I}(W^J).$$

Finally, in the local unramified case, G is cyclic, generated by the v-Frobenius σ_v and H the subgroup generated by the v'-Frobenius $\sigma_{v'} = \sigma_v^f$, where $f = (G : H)$.

Then $V = \oplus_{i=0}^{f-1} \sigma_v^i W$ and we can assume W is 1 dimensional, with basis x. Let $\sigma'_v x = \eta x$. For each solution ζ to $\zeta^f = \eta$ the element $\sum_{i=0}^{f-1} \zeta^{-i}\sigma^i x \in V$ is an eigenvector for σ_v with eigenvalue ζ. Hence

$$L_v(s, V) = \det(1 - \sigma_v q_v^{-s} \,|\, V) = \prod_{\zeta^f = \eta} (1 - \zeta q^{-s}) = (1 - \eta q^{-f} s)$$
$$= \det((1 - \eta q_v^{-s} \,|\, W) = L_{v'}(s, W).$$

1.6. Reciprocity and the relation between the two kinds of L-functions. Artin had quickly mastered[1] Takagi's recent work proving that "class fields" are just abelian extensions. Takagi had shown that for each level $\mathfrak{m}$ (see §1.1) there is an abelian extension $K_{\mathfrak{m}}$ of k with group $G_{K\mathfrak{m}/k}$ having the same invariants as, hence isomorphic to, the generalized ideal class group $\mathcal{C}_{\mathfrak{m}}$ and such that the way in which a prime ideal P of k decomposes in K is determined by the class of P mod $\mathfrak{m}$. From Takagi's decomposition law it follows that

$$\zeta_K(s) = \prod_{\chi} L(s, \chi) = \zeta_k(s) \cdot \prod_{\chi \neq 1} L(s.\chi), \tag{1.6.1}$$

where the first product is over all characters χ of $\mathcal{C}_{\mathfrak{m}}$. Artin realized that in the abelian case $K_{\mathfrak{m}}/k$ the simplest explanation for the existence of the two factorizations (1.5.5) and (1.6.1) of $\zeta_k(s)$ was that they be the same, and the simplest explanation for that would be the existence of a **canonical** isomorphism $\mathcal{C}_{\mathfrak{m}} \xrightarrow{\sim} G_{K_{\mathfrak{m}}/k}$ which, for each finite place v of k unramified in K, associated to the class of the prime P_v the Frobenius substitution σ_v. To prove this he had to show, for every element $a \in O_k$, such that $a \equiv 1 \bmod^{\times} \mathfrak{m}$

$$(a) = \prod_v P_v^{m_v} \quad \Rightarrow \quad \prod_v \sigma_v^{m_v} = 1. \tag{1.6.2}$$

It took him 4 years. Artin called (1.6.2) the reciprocity law, because every known explicit reciprocity law could be interpreted as (1.6.2) for some special Kummer extension K/k. For example, if q is a prime $\equiv 1$ mod 4, then for the extension $\mathbb{Q}(\sqrt{q})/\mathbb{Q}$), (1.6.2) implies that there is a non-trivial character χ of order 2 mod q such that $\chi(p) = (\frac{q}{p})$ for primes $p \neq q$. The only such character is $p \mapsto (\frac{p}{q})$. Hence, $(\frac{q}{p}) = (\frac{p}{q})$.

1.7. Brauer's theorem, meromorphicity.
A representation V of G and its character χ_V are called *monomial* if V is induced from a 1-dimensional representation of some subgroup of G, or in other words, V is a direct sum of 1-dimensional subspaces which are permuted transitively by G.

1.7.1. Theorem (R. Brauer). *Every character of a finite group G is a linear combination with integral coefficients of monomial characters* [**10**].

Brauer's main motivation for proving this was the fact that if the ψ_i are 1-dimensional characters of some subgroups H_i of G such that

$$\chi = \sum_i n_i \operatorname{Ind}_{H_i}^G \psi_i, \tag{1.7.2}$$

[1] He tells the story that, when as Post-doc in Göttingen, he asked Siegel if he could borrow Siegel's preprint of Takagi's paper, Siegel said yes, he could have it for 24 hours. Artin spent the time copying by hand the essential parts.

then

$$L(s,\chi) = \prod_i L(s,\psi_i)^{n_i}, \tag{1.7.3}$$

By the reciprocity law, the $L(s,\psi_i$ are classical abelian L-series, hence meromorphic on $\mathbb{C}$, and even entire for $\psi \neq 1$ (See §1.2). Hence Brauer's theorem implies

1.7.4. Corollary. *Every Artin L-series is meromorphic in the whole complex plane.*

1.7.5. Exercise. Show that if the trivial reresentation does not occur in χ, then one can cancel on the right side of (1.7.3) all of the terms in which ψ_i is the trivial character of H_i. (Write the hypothesis in the form $\langle\chi, 1_G\rangle_G = 0$ and use Frobenius reciprocity.)

If in (1.7.3) $n_i \geq 0$ for all i, and $\psi_i \neq 1$ for all i, then $L(s,\chi)$ is entire. However for most χ such an expression does not exist. Artin's conjecture that $L(s,\chi)$ is holomorphic for irreducible $\chi \neq 1$ is much deeper, implying a miraculous cancellation of the zeros of the $L(s,\psi_i)$ with $n_i < 0$ by those of those with $n_i > 0$ in (1.7.3).

1.8. Functional equation.

To get a neater functional equation, Artin defined local L-functions also for infinite places v of k. For such a v the decomposition group G_w of a place w of K above v is of order 1 or 2, hence has a unique generator, which we denote by σ_w, and we put

$$L_v(s,\chi) = (\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}))^{\chi(1)} = (2^{1-s}\sqrt{\pi}\Gamma(s))^{\chi(1)}, \text{ for } v \text{ complex}, \tag{1.8.1}$$

and

$$L_v(s,\chi) = \Gamma(\frac{s}{2})^{\frac{\chi(1)+\chi(\sigma_w)}{2}} \cdot \Gamma(\frac{s+1}{2})^{\frac{\chi(1)-\chi(\sigma_w)}{2}}, \text{ for } v \text{ real}. \tag{1.8.2}$$

Note that these L_v's depend only on the action of σ_w on V, i.e., only on V as G_w-module, just as was the case for finite v.

1.8.3. Exercise. Show that (1.5.7) holds for infinite v as well.

Let

$$\Lambda(s,V) := \prod_{\text{all } v} L_v(s,V) = \Gamma(\frac{s}{2})^a \cdot \Gamma(\frac{s+1}{2})^b L(s,V), \tag{1.8.4}$$

where $a = (r_2 + r_1^+)\dim V$ and $b = (r_2 + r_1^-)\dim V$, where r_1^+ (resp. r_1^-) is the number of real places of k such that the places of K above v are real (resp. complex). The three basic properties (1.5.2), (1.5.3) and (1.5.4) of $L(s,V)$ as function of V hold for $\Lambda(s,V)$ as well. This is trivial, given Exercise 1.8.3. For an abelian character ψ our definition of $\Lambda(s,\psi)$ is consistent with the definition in §1.2. Writing an arbitrary $L(s,\chi)$ in the form (1.7.2) and recalling the functional equation (1.2.3) for abelian L-functions, one sees that there are constants $B_\chi = \prod_i B_{\psi_i}^{n_i}$ and $C_\chi = \prod_i C_{\psi_i}^{n_i}$, such that

$$\Lambda(1-s,\chi) = C_\chi B_\chi^s \Lambda(s,\bar{\chi}). \tag{1.8.5}$$

Note that he B and C are uniquely determined by this equation, independent of the choice of expression (1.7.2), because $CB^s = 1$ for all s implies $B = C = 1$.

Exercise: Assuming (1.2.3) show that there exist unique constants $A_\chi > 0$ and $W_\chi \in \mathbb{C}^*$ such that $A_\chi = A_{\bar{\chi}}$ such that the functions

$$\xi(s.\chi) := A_\chi^{s/2}\Lambda(s,\chi) = A_\chi^{s/2}\Gamma(\frac{s}{2})^a \cdot \Gamma(\frac{s+1}{2})^b L(s,V), \tag{1.8.6}$$

satisfy

$$\xi(1-s,\chi) = W_\chi \cdot \xi(s,\bar{\chi}). \tag{1.8.7}$$

Show also that $|W_\chi| = 1$, and $W_{\bar{\chi}} = \bar{W}_\chi$. (Suggestion: Note that $L(\bar{s},\bar{\chi}) = \overline{L(s,\chi)}$ and consider the functional equation on the vertical line $\Re(s) = \frac{1}{2}$.) The constant W_χ is called an"Artin root number". Artin showed that

$$A_\chi = \frac{|d_k|^{\chi(1)}\mathbb{N}\mathfrak{f}(\chi)}{\pi^{[k:\mathbb{Q}]\chi(1)}}, \tag{1.8.8}$$

where $\mathfrak{f}(\chi)$ is an integral ideal of $\mathcal{O}_k$ involving only primes ramified in K called the *conductor* of χ.

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