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Integral and p-adic Refinements of the Abelian Stark Conjecture

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Introduction

In the 1970s and early 1980s, Stark [**St**] developed a remarkable Galois–equivariant conjectural link between the values at $s = 0$ of the first non–vanishing derivatives of the Artin L–functions $L_{K/k}(\rho, s)$ associated to a Galois extension K/k of number fields and a certain $\mathbb{Q}[\mathrm{Gal}(K/k)]$–module invariant associated to the group of global units of K. Stark's Main Conjecture should be viewed as a vast Galois–equivariant generalization of the unrefined, rational version of Dirichlet's class–number formula

$$\lim_{s\to 0}\frac{1}{s^r}\zeta_k(s) \in \mathbb{Q}^\times \cdot R_k\,,$$

in which the zeta function ζ_k is replaced by a Galois–equivariant L–function

$$\Theta_{K/k,S}(s) = \sum_{\rho\in\widehat{G}} L_{K/k,S}(\rho, s)\cdot e_{\check{\rho}}\,,$$

with values in the center of the group–ring $\mathcal{Z}(\mathbb{C}[\mathrm{Gal}(K/k)])$, the regulator R_k is replaced by a Galois–equivariant regulator with values in $\mathcal{Z}(\mathbb{C}[\mathrm{Gal}(K/k)])$, and the rank r of the group of units in k is replaced by the local rank function of the (projective) $\mathbb{Q}[\mathrm{Gal}(K/k)]$–module $\mathbb{Q}U_S$ of S–units in K.

In the 1970s and early 1980s, work of Stark, Tate, Gross, and Chinburg among others revealed not only the depth and importance of Stark's Main Conjecture for number theory (e.g. Chinburg's theory of multiplicative Galois Module Structure emerged from this context), but also the fact that an *integral refinement* of this statement, in the spirit of the integral Dirichlet class–number formula

$$\lim_{s\to 0}\frac{1}{s^r}\zeta_k(s) = -\frac{h_k}{w_k}\cdot R_k\,,$$

would have very far reaching applications to major unsolved problems in the field. In [**St IV**], Stark himself formulated such an integral refinement for *abelian extensions* K/k and their associated imprimitive L–functions $L_{K/k,S}(\chi, s)$ of *order of*

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vanishing at most 1 *at* $s = 0$. Roughly speaking, this integral statement predicts the existence of a special S–unit ε_K in K, which has remarkable arithmetic properties and, if evaluated against the Galois–equivariant regulator, it produces the value at $s = 0$ of the first derivative of the Galois–equivariant L–function $\Theta_{K/k,S}(s)$. Moreover, ε_K is unique and computable in terms of these special L–values.

In all the major instances in which this conjecture has been proved, the Stark units ε_K have turned out to be truly remarkable arithmetic objects. They are cyclotomic S–units or Gauss sums if $k = \mathbb{Q}$ (see [**St IV**] and [**Ta4**]), elliptic units if $k = \mathbb{Q}(\sqrt{-d})$, with $d \in \mathbb{Z}_{\geq 1}$ (see [**St IV**] and [**Ta4**]), and norms of torsion points of sign–normalized rank 1 Drinfeld modules if k is a function field (see [**H1**]). In each of these instances the construction of Stark's units ε_K is closely related to the solution of Hilbert's 12th problem for the respective base field k (i.e. the explicit generation of the abelian class–fields of the field k.) Tate [**Ta4**] showed that this is not at all coincidental – it turns out that, when non–trivial, the solution to Stark's Integral Conjecture would lead to an explicit generation of the abelian class–fields of the base field k by exponentials of special values of L–functions.

In the late 1980s and early 1990s, the emergence of Kolyvagin's theory of Euler Systems revealed a new interpretation of the known Stark units ε_K – they provide us with the only known non–trivial examples of Euler Systems of units: the Euler Systems of cyclotomic units, Gauss sums, elliptic units, and torsion points of rank 1 Drinfeld modules, leading to enlightening solutions of major problems in number theory – the Iwasawa Main Conjecture over $\mathbb{Q}$ and over quadratic imaginary fields, particular cases of the Birch–Swinnerton Dyer conjecture etc. Rubin [**Ru2**] showed that this is not a coincidence either – it turns out that, when non–trivial, the Stark units ε_K for various extensions K/k give rise to Euler Systems over a fixed field k.

Unfortunately, since in most cases the order of vanishing at $s = 0$ of the non–primitive L–functions $L_{K/k,S}(\chi, s)$ is strictly larger than 1, Stark's integral refinement of his Main Conjecture has a non–trivial output ε_K for a very limited class of abelian extensions K/k. This is why an integral refinement of Stark's Main Conjecture in its full generality is needed. Such a refinement is also expected to have the type of applications described in the previous paragraph for general extensions K/k. In 1994, Rubin [**Ru3**] formulated an integral refinement of the Main Conjecture for *abelian extensions* K/k and their associated imprimitive L–functions $L_{K/k,S}(\chi, s)$ *of arbitrary order of vanishing at* $s = 0$.

The main objectives of this paper are as follows: after introducing the necessary notations and definitions (see §1), we state Rubin's Conjecture and a related refinement of Stark's Main Conjecture due to the present author and discuss their links to the classical Integral Stark Conjecture (see §2); in §3 we discuss a series of applications of Rubin's Conjecture to the theory of Euler Systems, the construction of groups of special units and refined class-number formulas (Gras–type conjectures); in §4 we discuss a refinement of Rubin's Conjecture essentially due to Gross [**Gro1–3**] and interpret it in terms of special values of p–adic L–functions; in §5 [TO BE WRITTEN], we provide evidence in support of the Rubin-Stark and Gross Conjectures.

This introduction would be incomplete without mentioning the recent remarkable work of Burns, Flach, and their students and collaborators on the Equivariant Tamagawa Number Conjecture (ETNC), which has brought a wide variety of new and exciting ideas, techniques and interpretations to the subject of Stark's Conjectures. Building upon earlier work of Bloch–Kato and Fontaine–Perrin-Riou, Burns and Flach [**BF1–3**] have formulated the ETNC for L–functions associated to motives with (not necessarily abelian) coefficients. If restricted to Artin motives, this statement can also be viewed as an integral refinement of Stark's Main Conjecture for general (not necessarily abelian) extensions K/k. In the more restrictive case of Dirichlet motives, the ETNC implies (refinements of) the Rubin-Stark and Gross Conjectures discussed in this paper, as shown by Burns in [**Bu5**].

1. Notations and definitions

1.1. Notations. Let K/k be a finite, abelian extension of global fields of arbitrary characteristic and of Galois group $G := \mathrm{Gal}(K/k)$. We denote by $\widehat{G}$ the group of irreducible complex valued characters of G. Let μ_K be the group of roots of unity in K, $w_K := \mathrm{card}(\mu_K)$, and S and T two finite, nonempty sets of primes in k. For a finite extension K'/k, $S_{K'}$ and $T_{K'}$ will denote the sets of primes in K' dividing primes in S and T, respectively . For the moment, we require that the sets S and T satisfy the following set of hypotheses.

Hypotheses $(\mathbf{H}_0)$

1. S contains all the primes which ramify in K/k, and all the infinite primes of k in the case where k is a number field.
2. $T \cap S = \emptyset$.
3. There are no nontrivial elements in μ_K which are congruent to 1 modulo all the primes w in T_K.

The reader will note that the last hypothesis above is automatically satisfied in the function field case. In the number field case, it is satisfied if, for example, T contains either at least two primes of different residual characteristic or a prime whose corresponding residue field is large compared to the size of μ_K.

For a finite extension K'/k, $O_{K',S}$ will denote its ring of $S_{K'}$–integers, $U_{K',S} := O_{K',S}^{\times}$ is the group of $S_{K'}$–units in K', and $A_{K',S}$ the ideal–class group of $O_{K',S}$. For any such K', we also define the (S,T)–modified group of units and respectively ideal class–group as follows.

$$U_{K',S,T} := \{x \in U_{K',S} \mid x \equiv 1 \mod w, \forall w \in T_{K'}\} \ .$$

$$A_{K',S,T} := \frac{\{\text{fractional ideals of } O_{K',S} \text{ coprime to } T_{K'}\}}{\{x \cdot O_{K',S} \mid x \equiv 1 \mod w, \forall w \in T_{K'}\}} \ .$$

For simplicity, we will set $U_S := U_{K,S}$, $A_S := A_{K,S}$, $U_{S,T} := U_{K,S,T}$, and $A_{S,T} := A_{K,S,T}$. Since S_K and T_K are G–invariant, these groups are endowed with natural $\mathbb{Z}[G]$–module structures.

1.2. The G–equivariant L–function. For K/k, S, and T as above, and any χ in $\widehat{G}$, let $L_S(\chi, s)$ denote, as usual, the L–function associated to χ with Euler factors at primes in S removed, of the complex variable s. This is a complex valued function, holomorphic everywhere if χ is non–trivial, and holomorphic outside $s = 1$, with a pole of order 1 at $s = 1$ if χ is the trivial character. With the help of these L–functions, one can define

$$\Theta_S\,, \quad \Theta_{S,T} : \mathbb{C} \longrightarrow \mathbb{C}[G]\,,$$

$$\Theta_S(s) := \sum_{\chi \in \widehat{G}} L_S(\chi, s) \cdot e_{\chi^{-1}}\,, \quad \Theta_{S,T} := \prod_{v \in T} (1 - \sigma_v^{-1} \cdot (\mathrm{N}v)^{1-s}) \cdot \Theta_S(s)\,,$$

where σ_v and $\mathrm{N}v$ denote the Frobenius morphism in G and the cardinality of the residue field associated to v, and

$$e_{\chi^{-1}} := 1/|G| \sum_{\sigma\in G} \chi(\sigma)\cdot\sigma$$

is the idempotent element associated to χ^{-1} in $\mathbb{C}[G]$. The functions Θ_S and $\Theta_{S,T}$ are the so–called S–modified respectively (S,T)–modified G–equivariant L–function. If the group–ring $\mathbb{C}[G]$ is viewed in the obvious manner as a direct product of $|G|$ copies of $\mathbb{C}$, then the projections of Θ_S and $\Theta_{S,T}$ onto the various components of $\mathbb{C}[G]$ with respect to this product decomposition are holomorphic away from $s=1$ (holomorphic everywhere, respectively) as functions of the complex variable s. The values of $\Theta_{S,T}$ at non-positive integers satisfy the following remarkable integrality property.

THEOREM 1.2.1. *Under hypotheses* (H_0)*, one has*

$$\Theta_{S,T}(1-n)\in\mathbb{Z}[G]\,,$$

for all integers $n\geq 1$.

In the number field case, Theorem 1.2.1 was independently proved by Deligne–Ribet [**DR**], P. Cassou-Nogues [**CN**], and D. Barski [**Bar1**]. In fact, in [**DR**] it is proved that one has

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_{K,n})\cdot\Theta_S(1-n)\subseteq\mathbb{Z}[G]\,,\ \text{for all } n\in\mathbb{Z}_{\geq 1}.$$

Here, $\mu_{K,n}$ denotes the group of roots of unity in the maximal abelian extension $K^{(n)}$ of K of exponent dividing n, on which G acts via "lifts and n–powers" (This means that $\sigma*\zeta=\widetilde{\sigma}^n(\zeta)$, for all $\zeta\in\mu_{K,n}$ and $\sigma\in G$, where $\widetilde{\sigma}$ is an arbitrary lift of σ to $G(K^{(n)}/k)$. Note that $\mu_{K,1}=\mu_K$. A more familiar notation for $\mu_{K,n}$ might be $(\mathbb{Q}/\mathbb{Z}(n))^{G_K}$.) The reader will notice that hypothesis $(\mathrm{H}_0)3$ is equivalent to

$$\prod_{v\in T}(1-\sigma_v^{-1}\cdot(\mathrm{N}v)^n)\in\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_{K,n})\,,\ \text{for all } n\in\mathbb{Z}_{\geq 1}\,.$$

In fact, the following lemma, whose proof in characteristic 0 can be found in [**Co**], holds true in both characteristics 0 and p.

LEMMA 1.2.2. *Assume that the set of data* $(K/k,S_0)$ *satisfies hypothesis* (H_0)*–1. Then, for all* $n\in\mathbb{Z}_{\geq 1}$, $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_{K,n})$ *is generated as a* $\mathbb{Z}[G]$*–module by the elements*

$$\delta_T(1-n):=\prod_{v\in T}(1-\sigma_v^{-1}\cdot(Nv)^n)\,,$$

where T *runs through all the finite sets of primes in* k*, such that* $(K/k,S,T)$ *satisfies hypotheses* (H_0).

In the function field case, Theorem 1.2.1 is a direct consequence of Weil's theorem expressing the L–functions as alternating products of characteristic polynomials of the action of a geometric Frobenius morphism on various G–eigenspaces of ℓ–adic étale cohomology groups of the smooth projective curve associated to the top field K. By using this interpretation, one can show that, if q is the cardinality of the field of constants for the base field k, then there exists a polynomial $P_{S,T}(X)$ in $\mathbb{Z}[G][X]$, such that

$$\Theta_{S,T}(s)=P_{S,T}(q^{-s})$$

(see [**Ta4**] and §3.1 below.) Theorem 1.2.1 follows immediately from this equality. In what follows, in order to simplify notations we will set

$$\delta_T := \delta_T(0)\,,$$

for any set T as in Lemma 1.2.2.

In the context described above, Stark's Conjecture provides a link between the lead term in the Taylor expansion at $s = 0$ of the G–equivariant L–function $\Theta_{S,T}(s)$ and certain arithmetic invariants of the abelian extension K/k. In order to make this link precise, we will need to impose additional conditions on the sets of primes S and T. Let us fix an integer $r \geq 0$. We associate to r the following (extended) set of hypotheses to be satisfied by the set of data $(K/k, S, T, r)$.

Hypotheses $(\mathbf{H}_r)$

1. S contains all the primes which ramify in K/k, and all the infinite primes of k in the case where k is a number field.
2. $\mathrm{card}(S) \geq r + 1$.
3. S contains at least r distinct primes which split completely in K/k.
4. $T \cap S = \emptyset$.
5. There are no nontrivial elements in μ_K which are congruent to 1 modulo all the primes w in T_K.

The following Lemma shows how hypotheses (H_r) control the order of vanishing $\mathrm{ord}_{s=0}$ at $s = 0$ of the associated G–equivariant L–functions.

Lemma 1.2.3. *If $(K/k, S, T)$ satisfies hypotheses (H_r), then* $\mathrm{ord}_{s=0} L_S(\chi, s) \geq r$*, $\forall \chi \in \widehat{G}$ and, consequently,*

$$\mathrm{ord}_{s=0}\Theta_S(s) = \mathrm{ord}_{s=0}\Theta_{S,T}(s) \geq r\,.$$

Proof. This is a direct consequence of the following equality proved in [**Ta4**].

$$(1)\qquad \mathrm{ord}_{s=0} L_S(\chi, s) = \begin{cases} \mathrm{card}\{v \in S \mid \chi|_{G_v} = \mathbf{1}_{G_v}\}, & \text{for } \chi \neq \mathbf{1}_G \\ \mathrm{card}(S) - 1, & \text{for } \chi = \mathbf{1}_G \end{cases},$$

where G_v is the decomposition group of v in K/k and $\mathbf{1}_H$ denotes the trivial character of a group H. □

If the set of data $(K/k, S, T)$ satisfies hypotheses (H_r), we let

$$\Theta_{S,T}^{(r)}(0) := \lim_{s\to 0} \frac{1}{s^r}\Theta_{S,T}(s)$$

denote the coefficient of s^r in the Taylor expansion of $\Theta_{S,T}(s)$ at $s = 0$.

1.3. The G–equivariant regulator maps. The link between the analytic aspects of the picture (represented by the G–equivariant L–function described in the previous section) and its arithmetic aspects (represented by the $\mathbb{Z}[G]$–modules of units $U_{S,T}$ and ideal classes $A_{S,T}$) predicted by Stark's Conjecture is achieved via certain G–equivariant regulator maps. These will be defined in the present section.

Throughout this section we assume that the data $(K/k, S, T, r)$ satisfies hypotheses (H_r), for a fixed $r \in \mathbb{Z}_{\geq 0}$. We fix an r–tuple $V := (v_1, \ldots, v_r)$ of r distinct primes in S which split completely in K/k, and primes w_i in K, with w_i dividing v_i, for all $i = 1, \ldots, r$. Let $W := (w_1, \ldots, w_r)$. For all primes w in K, let $|\cdot|_w$ denote their associated metrics, canonically normalized so that the product formula holds. This means that for all $x \in K^\times$, we let

$$|x|_w := \begin{cases} (\mathrm{N}w)^{-\mathrm{ord}_w(x)}, & \text{if } w \text{ is finite} \\ |\sigma_w(x)|, & \text{if } w \text{ is infinite} \end{cases},$$

where σ_w denotes the unique embedding of K into $\mathbb{C}$ associated to w and $|\cdot|$ is the usual complex absolute value.

Throughout this paper, if M is a $\mathbb{Z}[G]$–module and R is a commutative ring with 1, then RM denotes the tensor product $R \otimes_{\mathbb{Z}} M$ endowed with the usual $R[G]$–module structure, $\widetilde{M}$ denotes the image of M via the canonical morphism

$$M \longrightarrow \mathbb{Q}M\,,$$

and $M^* := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ is the dual of M in the category of $\mathbb{Z}[G]$–modules.

Definition 1.3.1. *The G–equivariant regulator map associated to W is the unique $\mathbb{Q}[G]$–linear morphism*

$$R_W : \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T} \longrightarrow \mathbb{C}[G]\,,$$

such that, for all $u_1, \dots, u_r$ in $U_{S,T}$, we have

$$R_W(u_1 \wedge \dots \wedge u_r) := \det(-\sum_{\sigma \in G} \log|u_i^{\sigma^{-1}}|_{w_j} \cdot \sigma)\,,$$

where the determinant is taken over $\mathbb{C}[G]$, and $i, j = 1, \dots, r$.

Remark 1. If extended by $\mathbb{C}$–linearity the regulator R_W defined above induces a $\mathbb{C}[G]$–isomorphism (see [**Ru3**] for a proof)

$$R_W : \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T} \xrightarrow{\sim} \mathbb{C}[G]\,,$$

also denoted by R_W in what follows. By definition, the special value $\Theta_{S,T}^{(r)}(0)$ belongs to the $\mathbb{C}[G]$–submodule of $\mathbb{C}[G]$, consisting of all elements $x \in \mathbb{C}[G]$ which satisfy

$$e_\chi \cdot x = 0\,,$$

for all χ in $\widehat{G}$, such that $\mathrm{ord}_{s=0} L_S(\chi, s) > r$. Since R_W is a $\mathbb{C}[G]$–isomorphism, this implies that there exists a unique element

$$\varepsilon_{S,T} \in \mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\,,$$

such that the following equalities hold.

1. $R_W(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0)$.
2. $e_\chi \cdot \varepsilon_{S,T} = 0$ in $\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$, for all χ in $\widehat{G}$, such that $\mathrm{ord}_{s=0} L_S(\chi, s) > r$.

The above remark prompts us to give the following definition.

Definition 1.3.2. Assume that the set of data $(K/k, S, T, r)$ satisfies hypotheses (H_r). Let M be a $\mathbb{Z}[G]$–module. We define

$$\widetilde{M}_{r,S} := \{x \in \widetilde{M} \mid e_\chi \cdot x = 0 \text{ in } \mathbb{C}M, \text{ for all } \chi \in \widehat{G} \text{ such that } \mathrm{ord}_{s=0} L_S(\chi, s) > r\}\,.$$

2. The Conjectures

In this section, we state Stark's Conjecture ("over $\mathbb{Q}$") in the abelian situation outlined in §1, as well as its integral refinements ("over $\mathbb{Z}$") due to Rubin and the present author.

2.1. Stark's Conjecture "over $\mathbb{Q}$".

CONJECTURE $\mathrm{A}(K/k, S, T, r)$ (Stark). *If the set of data $(K/k, S, T, r)$ satisfies hypotheses* (H_r)*, then there exists a unique element*

$$\varepsilon_{S,T} \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\,,$$

such that the following equalities hold.

1. $R_W(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0)$.
2. $e_\chi \cdot \varepsilon_{S,T} = 0$ *in* $\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$ *, for all* χ *in* $\widehat{G}$*, such that* $\mathrm{ord}_{s=0} L_S(\chi, s) > r$.

A few remarks concerning this statement are in order.

Remark 1. The uniqueness part of the statement above is a direct consequence of Remark 1, §1.3. The (highly non–trivial) conjectural part is the statement that the unique element $\varepsilon_{S,T} \in (\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T})_{r,S}$ satisfying the regulator condition (1) and the vanishing condition (2) in Remark 1, §1.3, belongs in fact to the $\mathbb{Q}[G]$–submodule $(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T})_{r,S}$.

Remark 2. No doubt, the reader has noticed that the notation used for $\varepsilon_{S,T}$ does not capture its dependence of our choice of r–tuples $V := (v_1, \dots, v_r)$ and $W = (w_1, \dots, w_r)$ of primes in S and S_K, as in §1.3. It is indeed true that $\varepsilon_{S,T}$ depends on these choices. However, it turns out that this dependence is simple and, most importantly, if Conjecture A is true for one choice of V and W, then it is true for any other choice. To show this, let us first assume that V' and W' differ from V and W by a permutation $\tau \in \mathrm{Sym}_r$ and respectively an r–tuple $\sigma := (\sigma_1, \dots, \sigma_r) \in G^r$, i.e. $V' = (v_{\tau(1)}, \dots, v_{\tau(r)})$ and $W' = (v_{\tau(1)}^{\sigma_1}, \dots, v_{\tau(r)}^{\sigma_r})$. Then one can define a unique $\mathbb{Z}[G]$–linear isomorphism

$$\Phi_{\tau,\sigma} : \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T} \xrightarrow{\sim} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}\,,$$

by setting $\Phi_{\tau,\sigma}(u_1 \wedge \cdots \wedge u_r) := u_{\tau(1)}^{\sigma_1^{-1}} \wedge \cdots \wedge u_{\tau(r)}^{\sigma_r^{-1}}$, for all $u_1, \dots, u_r$ in $U_{S,T}$. Let $\Phi_{\tau,\sigma}$ denote the unique $\mathbb{C}$-linear extension of the above map to $\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$. If $\varepsilon_{S,T}$ and $\varepsilon'_{S,T}$ are the unique elements in $\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$ satisfying (1) and (2) in Remark 1, §1.3, for R_W and $R_{W'}$ respectively, then one can easily show that $\varepsilon'_{S,T} = \Phi_{\tau,\sigma}(\varepsilon_{S,T})$. This shows that if $\varepsilon_{S,T} \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$, then $\varepsilon'_{S,T} \in \mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$ and vice versa. Secondly, let us assume that S contains more than r primes which split completely, i.e. there are at least two choices V and V' which do not differ from one another via a permutation $\tau \in \mathrm{Sym}_r$. If $\mathrm{card}(S) > r + 1$, then Lemma 1.2.3 above implies that $\Theta_{S,T}^{(r)}(0) = 0$ and therefore Conjecture A is trivially true for $\varepsilon_{S,T} = 0$, for any choice of V and W. Now, let us assume that $S = \{v_1, \dots, v_{r+1}\}$, and v_i splits completely for all $i = 1, \dots, r+1$. Then Lemma 1.2.3 combined with the (S,T)–class–number formula of [**Ru3**] implies that

$$\Theta_{S,T}^{(r)}(0) = -\mathrm{card}(A_{k,S,T}) \cdot R_{k,S,T} \cdot e_{\mathbf{1}_G} ,$$

where $\mathbf{1}_G$ denotes the trivial character of G and $R_{k,S,T}$ is the Dirichlet regulator associated to the group of (S,T)–units $U_{k,S,T}$. The equalities above imply right away that in this case Conjecture A is true with

$$\varepsilon_{S,T} := \pm\frac{\mathrm{card}(A_{k,S,T})}{|G|^r} u_1 \wedge \cdots \wedge u_r ,$$

where $u_1, \dots, u_r$ is a $\mathbb{Z}$–basis for $U_{k,S,T}$ and the sign is uniquely determined by the choice of V and W. Based on these considerations and in order to simplify notations, we have dropped the dependence of $\varepsilon_{S,T}$ on the choice of V and W from the notation in the statement of Conjecture A and throughout this paper.

Remark 3. The reader familiar with [**St**] will notice that Stark's original formulation of his conjecture "over $\mathbb{Q}$" is quite different from ours. The main difference stems from the fact that while the statement above is given in a Galois–equivariant form, Stark's original conjecture was formulated in a character–by–character manner (i.e. for one L–function at a time). Yet another difference is marked by the fact that Stark deals with all the characters $\chi \in \widehat{G}$, while Conjecture A only deals with those characters χ whose associated L–functions have minimal order of vanishing r at $s = 0$. The statement presented here is essentially due to Tate [**Ta4**] and Rubin [**Ru3**] and, under the present hypotheses (H_r), is equivalent to Stark's original conjecture for L–functions of minimal order of vanishing r at $s = 0$ (see [**Ru3**] for a proof).

2.2. Rubin's integral refinement of Conjecture A. In this section, we state Rubin's integral refinement of Conjecture A. The main idea behind any integral refinement of Conjecture A is to construct an arithmetically meaningful $\mathbb{Z}[G]$–submodule (i.e. a G–equivariant lattice of rank which is not necessarily maximal) of the $\mathbb{Q}[G]$–module $\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$, which contains the element $\varepsilon_{S,T}$. We will first describe Rubin's construction of such a lattice. We are still working under hypotheses (H_r) for the set of data $(K/k, S, T, r)$. For any $(r-1)$–tuple

$$\Phi := (\phi_1, \dots, \phi_{r-1}) \in (U_{S,T}^*)^{r-1} ,$$

there exists a unique $\mathbb{Q}[G]$–linear morphism

$$\widetilde{\Phi} : \mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T} \longrightarrow \mathbb{Q}U_{S,T} ,$$

such that, for all $u_1, \dots, u_r$ in $U_{S,T}$, we have

$$\widetilde{\Phi}(u_1 \wedge \cdots \wedge u_r) = \sum_{k=1}^{r} (-1)^k \det(\phi_i(u_j))_{j \neq k} \cdot u_j .$$

In the last equality, the determinant in the k–th term of the sum is taken with respect to all $i = 1, \dots, r-1$ and all $j = 1, \dots, r$, such that $j \neq k$. Please note that since $U_{S,T}$ has no $\mathbb{Z}$–torsion (see hypothesis $(\mathrm{H}_r)5$), $U_{S,T}$ can be naturally viewed as a $\mathbb{Z}[G]$–submodule of $\mathbb{Q}U_{S,T}$. Consequently, the k–term sum above can be viewed without any ambiguity inside $\mathbb{Q}U_{S,T}$.

DEFINITION 2.2.1. Rubin's lattice $\Lambda_{S,T}$ consists of all elements ϵ in $\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}$, which satisfy the following properties.

1. $\tilde{\Phi}(\epsilon) \in U_{S,T}$, for all $\Phi := (\phi_1, \dots, \phi_{r-1}) \in (U_{S,T}^*)^{r-1}$.
2. $e_\chi \cdot \epsilon = 0$ in $\mathbb{C}[G]$, for all $\chi \in \widehat{G}$, such that $\mathrm{ord}_{s=0} L_S(\chi, s) > r$.

REMARK 1. It is immediate from Definition 2.2.1 that for $r = 0, 1$, we have

$$\Lambda_{S,T} = \begin{cases} (\widetilde{U_{S,T}})_{1,S}\,, & \text{if } r = 1 \\ \mathbb{Z}[G]_{0,S}\,, & \text{if } r = 0\,. \end{cases}$$

For a general $r \geq 1$, we have inclusions

$$|G|^n \cdot \Lambda_{S,T} \subseteq (\widetilde{\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}})_{r,S} \subseteq \Lambda_{S,T}\,,$$

for sufficiently large positive integers n. It is not difficult to show that if $U_{S,T}$ has finite projective dimension over $\mathbb{Z}[G]$, then the second inclusion above is an equality. However, as Rubin shows in [**Ru3**], the second inclusion above is in general strict. After tensoring with $\mathbb{Z}[1/|G|]$, we have an equality

$$\mathbb{Z}[1/|G|]\Lambda_{S,T} = (\widetilde{\mathbb{Z}[1/|G|]\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T}})_{r,S}\,,$$

which follows directly from the sequence of inclusions above. Also, since $U_{S,T}$ sits inside U_S with a finite index, we have equalities

$$\mathbb{Q}\Lambda_{S,T} = (\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_{S,T})_{r,S} = (\mathbb{Q}\bigwedge_{\mathbb{Z}[G]}^{r} U_S)_{r,S}\,.$$

CONJECTURE B$(K/k, S, T, r)$ (RUBIN). *If the set of data* $(K/k, S, T, r)$ *satisfies hypotheses* (H_r)*, then there exists a unique element* $\varepsilon_{S,T} \in \Lambda_{S,T}$*, such that*

$$R_W(\varepsilon_{S,T}) = \Theta_{S,T}^{(r)}(0)\,.$$

REMARK 2. In the case $r = 0$ Rubin's conjecture states that $\Theta_{S,T}(0) \in \mathbb{Z}[G]_{0,S}$. This statement is true and follows immediately if one sets $n = 1$ in Theorem 1.2.1 and applies equalities (1) above.

REMARK 3. In the case $r = 1$, Conjecture B is equivalent to Stark's integral refinement of Conjecture A for L–functions of order of vanishing 1 at $s = 0$, formulated in [**St IV**]. A proof of this equivalence can be found in [**Ta4**] and [**Ru3**] and it is a direct consequence of Remark 1 for $r = 1$ and Lemma 2.2.3 below.

At times, it is convenient to drop dependence on the auxiliary set T in conjecture B formulated above. This is why we will sometimes work with the following statement instead.

CONJECTURE $\mathrm{B}(K/k,S,r)$. *If the set of data $(K/k,S,r)$ satisfies hypotheses (H_r)1–3, then for all sets T such that the set of data $(K/k,S,T,r)$ satisfies hypotheses (H_r), there exists a unique element $\varepsilon_{S,T}\in\Lambda_{S,T}$, such that*

$$R_W(\varepsilon_{S,T})=\Theta_{S,T}^{(r)}(0)\,.$$

The following result proved in [**P4**] (see Proposition 5.3.1) shows that in the above conjecture it is sufficient to work with minimal sets T.

THEOREM 2.2.2. *Let $(K/k,S,T,r)$ and $(K/k,S,T',r)$ be two sets of data satisfying hypotheses (H_r), such that $T\subseteq T'$. Then*

$$B(K/k,S,T,r)\Longrightarrow B(K/k,S,T',r)\,.$$

In particular, this theorem shows that, in the case where $\mathrm{char}(k)=p>0$, it suffices to prove conjecture $B(K/k,S,T,r)$ for sets T of cardinality 1.

REMARK 4. In what follows, we will give a somewhat detailed description of the connection between Rubin's Conjecture and the classical conjectures of Brumer and Brumer–Stark. In the 1970s, Brumer stated the following conjecture, as a natural extension of the classical Theorem of Stickelberger for abelian extensions of $\mathbb{Q}$ (see [**Co**]) to abelian extensions of general number fields. The conjecture was later extended to global fields of arbitrary characteristic by Mazur and Tate (see [**Ta4**]).

CONJECTURE $\mathrm{Br}(K/k,S_0)$ (BRUMER). *Let us assume that $(K/k,S_0)$ satisfies hypothesis $(\mathrm{H}_0)1$. Then*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\cdot\Theta_{S_0}(0)\subseteq\begin{cases}\mathrm{Ann}_{\mathbb{Z}[G]}(A_K)\,, & \text{if } \mathrm{char}(k)=0\\ \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Pic}^0(K))\,, & \text{if } \mathrm{char}(k)>0\end{cases}$$

where A_K is the usual ideal–class group of the number field K and $\mathrm{Pic}^0(K)$ is the Picard group of equivalence classes of divisors of degree 0 of the function field K.

The $\mathbb{Z}[G]$–ideal $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\cdot\Theta_{S_0}(0)$ is called the Stickelberger ideal associated to $(K/k,S_0)$ and it generalizes the classical Stickelberger ideal defined for $k=\mathbb{Q}$ to the case of arbitrary base fields k. Stark (in the number field case) and Tate (in the function field case) formulated the following refinement of Brumer's conjecture, now commonly known as the Brumer–Stark Conjecture.

CONJECTURE $\mathrm{BrSt}(K/k,S_0)$ (BRUMER–STARK). *Let us assume that the set of data $(K/k,S_0)$ satisfies hypothesis $(\mathrm{H}_0)1$. Then,*

1. *If $\mathrm{char}(k)=0$ and I is a fractional ideal of K, there exists a unique element $\alpha_I\in(K^\times)_{0,S_0}$, such that*

$$w_K\Theta_{S_0}(0)\cdot I=(\alpha_I)$$

   and $K(\alpha_I^{1/w_K})/k$ is an abelian extension.

2. *If $\mathrm{char}(k)>0$ and D is a nonzero divisor of K, there exist unique $\alpha_D\in(K^\times)_{0,S_0}$ and $m_D\in\mathbb{Z}$, such that*

$$w_K\Theta_{S_0}(0)\cdot D=\mathrm{div}(\alpha_D)+m_D\cdot\sum_{w\in(S_0)_K}w$$

   and $K(\alpha_D^{1/w_K})/k$ is an abelian extension.

Showing that $BrSt(K/k, S_0)$ implies $Br(K/k, S_0)$ is an easy exercise based on Lemma 1.2.2 and on the following characterization of those elements $\alpha \in K^\times$, such that $K(\alpha^{1/w_K})/k$ is an abelian extension (see [**Ta4**] or [**P4**].)

LEMMA 2.2.3. *Let $(K/k, S_0)$ be as above, let $\alpha \in K^\times$, and set*

$$\operatorname{supp}(\alpha) := \{v \textit{ finite prime in } k \mid \operatorname{ord}_v(N_{K/k}(\alpha)) > 0\}\,.$$

Then, the following are equivalent.

1. *$K(\alpha^{1/w_K})/k$ is an abelian extension.*
2. *For all finite sets of primes T in k, such that $T \cap (S_0 \cup \operatorname{supp}(\alpha)) = \emptyset$ and $(K/k, S_0, T)$ satisfies hypotheses (H_0), there exists $\alpha_T \in K^\times$ such that*

$$\alpha^{\delta_T} = \alpha_T^{w_K}\,; \quad \alpha_T \equiv 1 \bmod^\times w\,, \textit{ for all } w \in T_K\,.$$

This lemma leads us to the following (S_0, T)–version of the Brumer–Stark Conjecture, which will be useful in our future considerations.

CONJECTURE $\mathrm{BrSt}(K/k, S_0, T)$. *Let us assume that $(K/k, S_0, T)$ satisfies hypothesis* (H_0). *Then,*

1. *If $\operatorname{char}(k) = 0$ and I is a fractional ideal of K coprime to T, there exists a unique element $\alpha_{I,T} \in (K^\times)_{0,S_0}$, such that $\alpha_{I,T} \equiv 1 \bmod^\times w$, for all $w \in T_K$, and*

$$\Theta_{S_0,T}(0) \cdot I = (\alpha_{I,T})\,.$$

2. *If $\operatorname{char}(k) > 0$ and D is a nonzero divisor of K coprime to T, there exist unique $\alpha_{D,T} \in (K^\times)_{0,S_0}$ and $m_D \in \mathbb{Z}$, such that $\alpha_{D,T} \equiv 1 \bmod^\times w$, for all $w \in T_K$, and*

$$\Theta_{S_0,T}(0) \cdot D = \operatorname{div}(\alpha_{D,T}) + m_D \cdot \sum_{w \in (S_0)_K} w\,.$$

Lemma 2.2.3 implies that proving conjecture $BrSt(K/k, S_0)$ is equivalent to proving conjectures $BrSt(K/k, S_0, T)$, for all T such that $(K/k, S_0, T)$ satisfies hypotheses (H_0).

REMARK. The reader will notice right away that, if $\operatorname{char}(k) > 0$ and $\operatorname{card}(S_0) > 1$, then $\mathrm{BrSt}(K/k, S_0)$ implies that the Stickelberger ideal associated to $(K/k, S_0)$ annihilates in fact the group $\operatorname{Pic}(K)$ of classes of K–divisors of arbitrary degree, which is much larger than $\operatorname{Pic}^0(K)$. Indeed this can be proved by taking divisor degrees in the equality displayed in Part 2. of the Brumer-Stark Conjecture $BrSt(K/k, S_0)$ stated above, and by noticing that, if $\operatorname{card}(S_0) > 1$, then

$$\deg(w_K \Theta_{S_0}(0) \cdot D) = \mathbf{1}_G(w_K \Theta_{S_0}) \cdot \deg(D) = 0$$

(see Lemma 1.2.3), for all K–divisors D. This implies that $m_D = 0$ and the desired result ensues by applying once again Lemma 1.2.3. If $\operatorname{card}(S_0) = 1$, then the Stickelberger ideal does not annihilate $\operatorname{Pic}(K)$, in general.

Rubin's Conjecture fully captures the annihilation phenomenon described in the Brumer–Stark Conjecture, as shown in the following theorem.

THEOREM 2.2.4. *Let us assume that the set of data* $(K/k, S_0)$ *satisfies hypothesis* $(\mathrm{H}_0)1$. *Then we have an equivalence*

$$BrSt(K/k, S_0) \Longleftrightarrow \left\{ \begin{array}{l} B(K/k, S_0 \cup \{v\}, 1)\,, \ \textit{for all primes } v \textit{ in } k, \\ v \textit{ split in } K/k, \ v \notin S_0. \end{array} \right\}$$

PROOF (sketch). In the case $\mathrm{char}(k) = 0$ this is a consequence of Lemma 2.2.3 above and the fact that each ideal–class in A_K has infinitely many representatives w, where w is a (finite) prime in K, dividing a prime v in k which splits completely in K/k (a consequence of Tchebotarev's density theorem).

In the case $\mathrm{char}(k) > 0$ the necessary ingredients for the proof are Lemma 2.2.3 and the fact that the group of K–divisor classes $\mathrm{Pic}(K)$ is generated by classes of prime divisors Π in K sitting above prime divisors π in k which are completely split in K/k. (A class–field theoretic proof of this fact can be found in [**H1**], p. 8.) □

2.3. A weaker integral refinement for Conjecture A. In our study of a base–change–type functoriality property for Conjecture B, we arrived at a slightly weaker integral refinement for Conjecture A (see [**P4**]). From a functorial point of view, this new statement seems to be more natural than Conjecture B. In this section, we state this refinement and describe its links to Conjecture B.

Let us fix $(K/k, S, T, r)$, satisfying hypotheses (H_r). As in [**P4**], we have an exact sequence of $\mathbb{Z}[G]$–modules

$$(2) \qquad 0 \longrightarrow U_{S,T} \longrightarrow U_S \xrightarrow{\mathrm{res}_T} \oplus_{w\in T_K} F(w)^\times \longrightarrow A_{S,T} \longrightarrow A_S \longrightarrow 0\,,$$

where A_S denotes the ideal class group of the ring of S–integers $O_{K,S}$ in K, $F(w)$ is the residue field of w, and res_T is the direct sum of the residual morphisms into $F(w)^\times$ for all $w \in T_K$. If we denote by $\mathcal{I}$ the image of res_T, and take duals in the category of $\mathbb{Z}[G]$–modules, we obtain the following exact sequence of abelian groups

$$0 \longrightarrow U_S^* \longrightarrow U_{S,T}^* \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathcal{I}, \mathbb{Z}[G])\,.$$

Since $\mathcal{I}$ is finite, $\mathrm{Ext}^1_{\mathbb{Z}[G]}(\mathcal{I}, \mathbb{Z}[G])$ is finite and consequently U_S^* can be viewed (via the usual restriction map) as a subgroup of finite (in general non–trivial) index in $U_{S,T}^*$. With this observation in mind, one can define a new lattice $\Lambda'_{S,T}$ inside $\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]} U_{S,T}$ as follows.

DEFINITION 2.3.1. The lattice $\Lambda'_{S,T}$ is the set of all elements ϵ in $\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]} U_{S,T}$, which satisfy the following properties.

1. $\tilde{\Phi}(\epsilon) \in U_{S,T}$, for all $\Phi := (\phi_1, \ldots, \phi_{r-1}) \in (U_S^*)^{r-1}$.
2. $e_\chi \cdot \epsilon = 0$ in $\mathbb{C}[G]$, for all $\chi \in \widehat{G}$, such that $\mathrm{ord}_{s=0} L_S(\chi, s) > r$.

Obviously, since U_S^* sits inside $U_{S,T}^*$ with a finite, in general non–trivial) index, Rubin's lattice $\Lambda_{S,T}$ sits inside the new lattice $\Lambda'_{S,T}$ with a finite (in general non–trivial) index. We are now ready to formulate the new integral refinement for Conjecture A mentioned in the introduction.

CONJECTURE C$(K/k,S,r)$. *Assume that the set of data $(K/k,S,r)$ satisfies hypotheses* (H_r)*1–3. Then, for all sets T such that the set of data $(K/k,S,T,r)$ satisfies hypotheses* (H_r)*, there exists a unique $\varepsilon_{S,T}\in\Lambda'_{S,T}$, such that*

$$R_W(\varepsilon_{S,T})=\Theta^{(r)}_{S,T}(0)\,.$$

Conjecture C satisfies the following remarkable base–change property. Let us assume that $(K/k,S,1)$ satisfies hypotheses (H_1)1–3, and let k' be an intermediate field, $k\subseteq k'\subseteq K$, and let $r:=[k':k]$. Then, it is obvious that $(K/k',S_{k'},r)$ satisfies hypotheses (H_r)1–3. In [**P4**] we prove the following.

THEOREM 2.3.2 (base–change for Conjecture C). *If conjecture $C(K/k,S,1)$ is true, then conjecture $C(K/k',S_{k'},r)$ is also true.*

It would be highly desirable to prove a similar base–change property for Conjecture B$(K/k,S,r)$. At this time, the techniques developed in [**P4**] have only led us to showing that, if $(K/k,S,T,1)$ satisfies hypotheses (H_1), then

$$B(K/k,S,T,1)\Longrightarrow B(K/k,S_{k'},T_{k'},r)\,.$$

This is obviously a much weaker result than the implication

$$B(K/k,S,r)\Longrightarrow B(K/k',S_{k'},r)\,,$$

as the $\mathrm{Gal}(k'/k)$–equivariant sets $T_{k'}$ are not minimal in general. However, in [**P4**] we also proved a comparison theorem which links Conjectures B and C and consequently leads to a base–change result for Conjecture B, if additional requirements are met. In what follows, we briefly describe this result.

If R is a subring of $\mathbb{Q}$ (e.g. $R=\mathbb{Q}$, $R=\mathbb{Z}[1/|G|]$, $R=\mathbb{Z}_{(\ell)}$ – the localization of $\mathbb{Z}$ at ℓ, for ℓ prime), then $RB(K/k,S,T,r)$ denotes the statement in Conjecture B$(K/k,S,T,r)$ with the lattice $\Lambda_{S,T}$ replaced by $R\Lambda_{S,T}\subseteq\mathbb{Q}\bigwedge^r_{\mathbb{Z}[G]}U_{S,T}$. We give a similar meaning to RB$(K/k,S,r)$, RC$(K/k,S,r)$, etc. In particular, we have the following equivalences.

(3)
$$\begin{gathered}A(K/k,S,T,r)\Longleftrightarrow\mathbb{Q}B(K/k,S,T,r)\Longleftrightarrow\mathbb{Q}B(K/k,S,r)\Longleftrightarrow\mathbb{Q}C(K/k,S,r)\\ B(K/k,S,T,r)\Longleftrightarrow\mathbb{Z}_{(\ell)}B(K/k,S,T,r)\,,\ \text{for all }\ell\text{ prime.}\end{gathered}$$

The first equivalence in the top row is a consequence of the last equality in Remark 1, §2.2. The second equivalence in the top row is a consequence of the fact that, for all T as above, the product

$$\delta_T:=\prod_{v\in T}(1-\sigma_v^{-1}\cdot\mathrm{N}v)$$

is invertible in $\mathbb{Q}[G]$. This implies that, for any T and T' such that $(K/k,S,T,r)$ and $(K/k,S,T',r)$ satisfy hypotheses (H_r), the unique elements

$$\varepsilon_{S,T}\,,\ \varepsilon_{S,T'}\in(\mathbb{C}\bigwedge^r_{\mathbb{Z}[G]}U_S)_{r,S}$$

satisfying $R_W(\varepsilon_{S,T})=\Theta^{(r)}_{S,T}(0)$ and $R_W(\varepsilon_{S,T'})=\Theta^{(r)}_{S,T'}(0)$, also satisfy the equality

$$\varepsilon_{S,T}=\delta_T\cdot\delta_{T'}^{-1}\cdot\varepsilon_{S,T'}\,.$$

Therefore, $\varepsilon_{S,T} \in (\mathbb{Q} \overset{r}{\underset{\mathbb{Z}[G]}{\wedge}} U_S)_{r,S}$ if and only if $\varepsilon_{S,T'} \in (\mathbb{Q} \overset{r}{\underset{\mathbb{Z}[G]}{\wedge}} U_S)_{r,S}$. Finally, the last equivalence in the top row is a direct consequence of the equality $\mathbb{Q}U_S^* = \mathbb{Q}U_{S,T}^*$.

Let μ_K be the group of roots of unity in K, endowed with the usual $\mathbb{Z}[G]$–module structure. For every prime number ℓ, let $\mu_K^{(\ell)}$ be the ℓ–Sylow subgroup of μ_K. We remind the reader that a $\mathbb{Z}[G]$–module M is called G–cohomologically trivial if

$$\widehat{\mathrm{H}}^i(H, M) = 0\,,$$

for all $i \in \mathbb{Z}$ and all subgroups $H \subseteq G$. Here $\widehat{\mathrm{H}}^i(H, M)$ denotes the i–th Tate cohomology group of H with coefficients in M. The following comparison theorem was proved in [**P4**].

Theorem 2.3.3. *Assume that the set of data $(K/k, S, r)$ satisfies hypotheses (H_r)1–3. Then, for all prime numbers ℓ, the following hold true.*

1. $\mathbb{Z}_{(\ell)}B(K/k, S, r) \Longrightarrow Z_{(\ell)}C(K/k, S, r)$.
2. *If $\mu_K^{(\ell)}$ is G–cohomologically trivial or $r = 1$, then*

$$\mathbb{Z}_{(\ell)}B(K/k, S, r) \Longleftrightarrow Z_{(\ell)}C(K/k, S, r)\,.$$

Since μ_K is G–cohomologically trivial if and only if $\mu_K^{(\ell)}$ is cohomologically trivial, for all prime numbers ℓ, and G–cohomological triviality is automatic at prime numbers ℓ which do not divide $|G|$, Theorem 2.3.3 implies the following.

Corollary 2.3.4. *If the set of data $(K/k, S, r)$ satisfies hypotheses (H_r)1–3, then the following hold true.*

1. $\mathbb{Z}[1/|G|]B(K/k, S, r) \Longleftrightarrow \mathbb{Z}[1/|G|]C(K/k, S, r)$.
2. *If μ_K is G–cohomologically trivial or $r = 1$, then*

$$B(K/k, S, r) \Longleftrightarrow C(K/k, S, r)\,.$$

In particular, Corollary 2.3.4 combined with Theorem 2.3.2 implies the desired base–change property for conjecture $\mathrm{B}(K/k, S, r)$, under the additional hypothesis that μ_K is G–cohomologically trivial. The following criterion (see Lemma 5.4.4 in [**P4**]) shows that while in the case $\mathrm{char}(k) = 0$ cohomological triviality for μ_K is a very rare event, in the case $\mathrm{char}(k) > 0$ it is satisfied for very large classes of abelian extensions K/k of a given base field k.

Lemma 2.3.5.

1. *If ℓ is odd or $\mathrm{char}(k) \neq 0$, then $\mu_K^{(\ell)}$ is G–cohom. trivial if and only if*

$$\ell \nmid w_K \ \textit{or}\ \ell \nmid [K : k(\mu_K^{(\ell)})]\,.$$

2. *If $\mathrm{char}(k) = 0$, then $\mu_K^{(2)}$ is G–cohom. trivial if and only if*

$$2 \nmid [K : k(\mu_K^{(2)})] \ \textit{and}\ \begin{Bmatrix} k \cap \mathbb{Q}(\mu_K^{(2)}) \text{ is not a (totally) real field} \\ \text{in the case } \mu_K^{(2)} \neq \mu_k^{(2)} \end{Bmatrix}.$$

In particular, Lemma 2.3.5 implies that, for any odd prime p and $n \geq 1$, $\mathbb{Q}(\mu_{p^n})/\mathbb{Q}$ satisfies G–cohomological triviality for $\mu_K^{(\ell)}$ for all odd primes ℓ but not for $\ell = 2$. On the other hand, if $\mathrm{char}(k) = p > 0$ and K is the compositum of an abelian p–power degree extension and an arbitrary finite constant field extension of

k, then the lemma implies that μ_K is G–cohomologically trivial (see §3.1 below). As it turns out, for a given global field k of characteristic $p > 0$, the compositum of all the finite abelian extensions K/k obtained this way sits inside the maximal abelian extension $k^{\rm ab}$ with a "quasi–finite index" (in a sense to be made more precise in §3.1 below.) Consequently, in characteristic $p > 0$, conjectures B and C are equivalent for a large class of extensions K/k.

3. Applications of Rubin's Conjecture

In this section we will give applications of Rubin's Conjecture to the theory of Euler Systems (following [**Ru2**], [**Ru3**] and [**Ru4**]) and the theory of special units and Gras–type conjectures (following [**Ru3**] and [**P3**]) in quite general contexts.

3.1. Euler Systems. In order to simplify matters, in what follows, k will denote a fixed totally real number field or characteristic p function field. Let T be a fixed, finite, nonempty set of non–archimedean (finite) primes in k, containing at least two primes of different residual characteristics, if $\mathrm{char}(k) = 0$. We let S_∞ denote the set of all archimedean (infinite) primes in k, if $\mathrm{char}(k) = 0$, and a fixed, finite, non–empty set of primes in k, with $T \cap S_\infty = \emptyset$, if $\mathrm{char}(k) = p$. We fix once and for all k, S_∞, and T as above, and fix a separable closure k^{sep} of k.

Definition 3.1.1. Let $\mathcal{K}$ be the set of all fields K, with $k \subseteq K \subseteq k^{sep}$, such that K/k is a finite abelian extension unramified at primes in T, totally split at primes in S_∞, and of nontrivial conductor $\mathfrak{f}_{K/k}$.

Definition 3.1.2. An Euler System of S_∞–units for $\mathcal{K}$ is a collection $(\kappa_L)_{L\in\mathcal{K}}$, with $\kappa_L \in U_{L,S_\infty}$, such that, for all $L' \subseteq L$ in $\mathcal{K}$, we have

$$\mathrm{Norm}_{L/L'}(\kappa_L) = \left(\prod_{v \mid \mathfrak{f}_{L/k}\,,\, v \nmid \mathfrak{f}_{L'/k}} (1 - \sigma_v^{-1})\right) \cdot \kappa_{L'}\,,$$

where σ_v is the Frobenius morphism associated to v in $G(L'/k)$.

In this section, we follow [**Ru3**] to show how one can construct non–trivial Euler Systems of S_∞–units for $\mathcal{K}$, assuming that Rubin's Conjecture holds true. For every $K \in \mathcal{K}$, let $S_K := S_\infty \cup \{\mathfrak{p} \mid \mathfrak{p} \text{ prime in } k\ ,\ \mathfrak{p} | \mathfrak{f}_{K/k}\}$. We also let $r := \mathrm{card}(S_\infty)$. Then, the choices we have made force the sets of data $(K/k, S_K, T, r)$, for all $K \in \mathcal{K}$, to satisfy hypotheses (H_r). *Throughout this section, we work under the assumption that Rubin's Conjecture $B(K/k, S_K, T, r)$ is true, for all $K \in \mathcal{K}$.* For every $K \in \mathcal{K}$, we let $\varepsilon_K := \varepsilon_{K/k,S_K,T}$ be the (unique) element in Rubin's lattice Λ_K associated to $(K/k, S_K, T, r)$, verifying conjecture $B(K/k, S_K, T, r)$. Also, let $U_K := U_{K,S,T}$ and

$$U_K^* := \mathrm{Hom}_{\mathbb{Z}[G(K/k)]}(U_K, \mathbb{Z}[G(K/k)])\,,$$

for all $K \in \mathcal{K}$.

Let $K, K' \in \mathcal{K}$, such that $K' \subseteq K$. Then, we have norm maps

$$U_K \xrightarrow{N_{K/K'}} U_{K'}\,, \qquad U_K^* \xrightarrow{N^*_{K/K'}} U^*_{K'}\,,$$

where $N_{K/K'}$ is the usual norm from $K^\times$ down to $(K')^\times$ restricted to U_K, and $N^*_{K/K'}$ is defined as follows. For every $\phi \in U_K^*$, we let

$$N^*_{K/K'}(\phi) := \frac{1}{[K:K']} \cdot \pi_{K/K'} \circ \phi \circ i_{K/K'}\,,$$

where $\pi_{K/K'} : \mathbb{Z}[G(K/k)] \longrightarrow \mathbb{Z}[G(K'/k)]$ is the canonical projection map induced by restriction at the level of Galois groups and $i_{K/K'} : U_{K'} \longrightarrow U_K$ is the inclusion map. The factor $1/[K:K']$ in the above definition is justified by the fact that for all $\phi \in U_K^*$ and all $u \in U_{K'}$, one has

$$\phi(u) \in \mathbb{Z}[G(K/k)]^{G(K/K')} = N_{G(K/K')} \cdot \mathbb{Z}[G(K/k)]\,,$$

where $N_{G(K/K')} := \sum_{\sigma \in G(K/K')} \sigma$. Consequently,

$$\pi_{K/K'} \circ \phi \circ i_{K/K'}(u) \in [K:K'] \cdot \mathbb{Z}[G(K'/k)]\,,$$

therefore

$$\frac{1}{[K:K']} \cdot \pi_{K/K'} \circ \phi \circ i_{K/K'}(u) \in \mathbb{Z}[G(K'/k)]\,.$$

However, a more conceptual justification of the definition of $N^*_{K/K'}$ can be given as follows. Since U_K and $U_{K'}$ have no $\mathbb{Z}$–torsion (a consequence of properties satisfied by the set T), we have canonical abelian group isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}}(U_K, \mathbb{Z}) \xrightarrow{\sim} U_K^*\,, \quad \mathrm{Hom}_{\mathbb{Z}}(U_{K'}, \mathbb{Z}) \xrightarrow{\sim} U_{K'}^*\,,$$

which associate to every $\psi \in \mathrm{Hom}_{\mathbb{Z}}(U_K, \mathbb{Z})$ the element $\widehat{\psi} \in U_K^*$, defined by

$$\widehat{\psi}(u) := \sum_{\sigma \in G(K/k)} \psi(u^{\sigma^{-1}}) \cdot \sigma\,,$$

for all $u \in U_K$, and similarly at the K'–level. As the reader can easily check, $N^*_{K/K'}$ defined above is the unique map which makes the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(U_K, \mathbb{Z}) & \xrightarrow{\mathrm{res}_{K/K'}} & \mathrm{Hom}_{\mathbb{Z}}(U_{K'}, \mathbb{Z}) \\ \downarrow \wr & & \downarrow \wr \\ U_K^* & \xrightarrow{N^*_{K/K'}} & U_{K'}^* \end{array}$$

commutative, where $\mathrm{res}_{K/K'}$ is the usual restriction of a function defined on U_K to its subgroup $U_{K'}$. Interpreting $N^*_{K/K'}$ in terms of the commutative diagram above proves to be beneficial from yet another point of view, made explicit below.

Lemma 3.1.2. *Under the assumptions and notations introduced above, the maps*

$$N^*_{K/K'} : U_K^* \longrightarrow U_{K'}^*$$

are surjective, for all $K, K' \in \mathcal{K}$.

Proof. It follows at once from Galois theory and the way T was chosen that the quotient group $U_K/U_{K'}$ has no $\mathbb{Z}$–torsion. This implies that the restriction map $\mathrm{res}_{K/K'}$ is surjective. Since the diagram above is commutative, $N^*_{K/K'}$ is surjective as well. □

We let $(N^*_{K/K'})^{(r-1)} := (N^*_{K/K'}, \ldots, N^*_{K/K'})$, viewed as (surjective) norm maps

$$(N^*_{K/K'})^{(r-1)} : (U_K^*)^{(r-1)} \longrightarrow (U_{K'}^*)^{(r-1)}\,,$$

at the level of direct sums of duals of groups of units. The following is proved in [**Ru3**].

Proposition 3.1.3. *For all* $\Phi = (\Phi_L)_{L\in\mathcal{K}} \in \varprojlim_{L\in\mathcal{K}} (U_L^*)^{(r-1)}$*, and all* $K, K' \in \mathcal{K}$*, such that* $K' \subseteq K$*, one has the following equality in* $U_{K'}$

$$N_{K/K'}(\widetilde{\Phi}_K(\varepsilon_K)) = \left(\prod_{v\in S_K\setminus S_{K'}} (1-\sigma_v^{-1})\right) \cdot \widetilde{\Phi}_{K'}(\varepsilon_{K'})\,,$$

where the projective limit is taken with respect to the norm maps $(N^*_{\cdot/\cdot})^{(r-1)}$ *and* σ_v *denotes the Frobenius morphism of* $v \in S_K \setminus S_{K'}$ *inside* $G(K'/k)$.

Proof (sketch). This is a direct consequence of the uniqueness of ε_K, for all $K \in \mathcal{K}$, and the following functoriality property of G–equivariant L–functions.

$$\begin{aligned}\pi_{K/K'}(\Theta_{K/k,S_K,T}(s)) &= \Theta_{K'/k,S_K,T}(s)\\ &= \left(\prod_{v\in S_K\setminus S_{K'}} (1-\sigma_v^{-1}\cdot \mathrm{N}v^{-s})\right)\cdot \Theta_{K'/k,S_{K'},T}(s)\,,\end{aligned}$$

for all $s \in \mathbb{C}$, where $\pi_{K/K'} : \mathbb{C}[G(K/k)] \longrightarrow \mathbb{C}[G(K'/k)]$ is the canonical $\mathbb{C}$–linear projection induced by the restriction map at the level of Galois groups. □

For every $K \in \mathcal{K}$, let $I(G(K/k))$ be the augmentation ideal in the integral group ring $\mathbb{Z}[G(K/k)]$. The projections $\pi_{K/K'}$ induce surjective maps

$$\pi_{K/K'} : I(G(K/k)) \longrightarrow I(G(K'/k))$$

at the level of augmentation ideals, for all $K' \subseteq K$ in $\mathcal{K}$.

Proposition 3.1.4. *The following hold true.*

1. *For all* $K \in \mathcal{K}$*,* $\eta_K \in I(G(K/k))$*, and* $\Phi_K \in (U_K^*)^{(r-1)}$*, we have*

$$\eta_K \cdot \widetilde{\Phi}_K(\varepsilon_K) \in U_{K,S_\infty,T}\,.$$

2. *Let* $\eta = (\eta_L)_{L\in\mathcal{K}} \in \varprojlim_{L\in\mathcal{K}} I(G(L/k))$ *and* $\Phi = (\Phi_L)_{L\in\mathcal{K}} \in \varprojlim_{L\in\mathcal{K}} (U_L^*)^{(r-1)}$*, and let* $K' \subseteq K$ *in* $\mathcal{K}$*. Then*

$$N_{K/K'}(\eta_K \cdot \widetilde{\Phi}_K(\varepsilon_K)) = (\prod_{v\in S_K\setminus S_{K'}} (1-\sigma_v^{-1}))\cdot(\eta_{K'}\cdot \widetilde{\Phi}_{K'}(\varepsilon_{K'}))\,,$$

viewed as an equality in $U_{K',S_\infty,T}$.

Proof (sketch). Statement (2) is a direct consequence of Proposition 4.1.3. Statement (1) follows from the fact that for all $K \in \mathcal{K}$ and all Φ_K, one has

$$\Phi_K(\varepsilon_K) \in (U_K)_{r,S_K}\,,$$

combined with the observation that

$$I(G(K/k))\cdot (U_K)_{r,S_K} \subseteq U_{K,S_\infty,T}\,.$$

□

Definition 3.1.2 and Proposition 3.1.4 lead us to the following.

Corollary 3.1.5. *If Rubin's Conjecture $B(K/k, S_K, T, r)$ holds true for all $K \in \mathcal{K}$, then for all $\eta = (\eta_L)_{L\in\mathcal{K}}$ and $\Phi = (\Phi_L)_{L\in\mathcal{K}}$ as in Proposition 3.1.4(2),*

$$(\eta_L \cdot \widetilde{\Phi}_L(\varepsilon_L))_{L\in\mathcal{K}}$$

is an Euler System of S_∞–units for $\mathcal{K}$.

Examples and Remarks. In the case $k = \mathbb{Q}$, the general construction outlined above leads to a slightly modified version of the classical Euler Systems of cyclotomic units (see [**Ru1**] and [**Ru2**]). Indeed, in this case S_∞ consists of the unique archimedean prime in $\mathbb{Q}$, so $r = 1$ and therefore the maps $\widetilde{\Phi}_K$ are not present. For a given T as above, the class $\mathcal{K}$ is dominated by $(\mathbb{Q}(\zeta_m)^+)_m$ with $m \in \mathbb{Z}_{\geq 3}$, m coprime to T. Therefore, for each $K \in \mathcal{K}$, $K \subseteq \mathbb{Q}(\zeta_m)^+$ for some m, for example $m = \mathfrak{f}_{K/\mathbb{Q}}$. Proposition 3.1.3 and comments after the [??? in §5] imply that, if we set $m = \mathfrak{f}_{K/\mathbb{Q}}$, then

$$\varepsilon_K = N_{\mathbb{Q}(\zeta_m)+/K}((\delta_{T,m} \cdot \epsilon_m)^{1/2}),$$

where ϵ_m denotes the cyclotomic element $(1-\zeta_m)(1-\zeta_m^{-1})$ in $\mathbb{Q}(\zeta_m)^+$ and $\delta_{T,m} = \prod_{v\in T}(1-\sigma_v^{-1}\cdot Nv)$ in $\mathbb{Z}[G(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})]$. Therefore, for any $\eta = (\eta_K)_{K\in\mathcal{K}}$ as above, the Euler System given by Proposition 4.1.4 in this case is uniquely determined via norm maps by the norm–coherent cyclotomic elements

$$\eta_{\mathbb{Q}(\zeta_m)^+} \cdot (\delta_{T,m} \cdot \epsilon_m)^{1/2}, \quad m \in \mathbb{Z}_{\geq 3}, \quad m \text{ coprime to } T.$$

Of course, we could have performed the same type of constructions, under the assumption that k is a totally imaginary field, the class $\mathcal{K}$ consists of arbitrary abelian extensions K of k of non–trivial conductor and Rubin's Conjecture $B(K/k, S_K, T, r)$ is true for all $K \in \mathcal{K}$. If restricted to the case where k is a quadratic imaginary field, one arrives this way at slightly modified versions of the classical Euler Systems of elliptic units (see [**Ru4**]).

The classical Euler Systems of Gauss sums (see [**Ru5**]) can be recovered very much in the same way, as a particular case of the situation where k is totally real, $\mathcal{K}$ consists of CM abelian extensions K of k, and appropriate choices for the sets S_K containing a fixed number r of non–archimedean primes of k which split completely in K/k.

In the case where $\text{char}(k) = p$ and $\text{card}(S_\infty) = 1$, the above construction leads to slight alterations of the Euler Systems of torsion points of rank one sign–normalized Drinfeld Modules constructed in [**FX**].

However, in all the classical Euler system constructions enumerated above, the order of vanishing of the associated L–functions is 1. If proved to be true, Rubin's Conjecture should be viewed as the source of non–trivial Euler Systems of units in the arbitrary order of vanishing case. It can be shown that the weaker conjecture C (see §2.3 above) has the same consequences in this context.

3.2. Groups of Special Units. Gras–type Conjectures. We work with the notations and under the assumptions of §4.1. We fix a field $K \in \mathcal{K}$. Our goal is to construct a group of special units $\mathcal{E}_{K,S_\infty,T}$ sitting inside $U_{K,S_\infty,T}$ as a $\mathbb{Z}[G(K/k)]$–submodule of finite index and satisfying an analogue of the classical Gras Conjecture for the group of cyclotomic units of a real cyclotomic field $\mathbb{Q}(\zeta_m)^+$. We follow [**P3**], where the construction of $\mathcal{E}_{K,S_\infty,T}$ and proofs of the Gras Conjectures were given in detail in the function field context, as consequences of the fact that Rubin's

Conjecture is true up to primes dividing the order of the Galois group $G(K/k)$ (see §5 ???). The constructions and proofs given in [**P3**] carry over almost word for word to the present more general context. We remind the reader that our main assumption is that conjecture $B(L/k, S_L, T, r)$ holds true for all $L \in \mathcal{K}$.

Definition 3.2.1. Let $h_{k,S_\infty,T} := \mathrm{card}(A_{k,S_\infty,T})$. The group of special units $\mathcal{E}_{K,S_\infty,T}$ is defined to be the $\mathbb{Z}[G(K/k)]$–submodule of $U_{K,S_\infty,T}$ generated by

$$u^{h_{k,S_\infty,T}}, \quad \eta_{K'} \cdot \widetilde{\Phi}_{K'}(\varepsilon_{K'}),$$

for all units $u \in U_{k,S_\infty,T}$, fields $K' \in \mathcal{K}$ with $K' \subseteq K$, $\eta_{K'} \in I(G(K'/k))$, and $\Phi_{K'} \in (U^*_{K'})^{(r-1)}$.

Proceeding as in [**P3**], one can eliminate T from the constructions above and define what we call the group of Stark S_∞–units of K (or plainly Stark units, if $\mathrm{char}(K) = 0$) as follows.

Definition 3.2.2. The group of Stark S_∞–units $\mathcal{E}_{K,S_\infty}$ of K is defined to be the $\mathbb{Z}[G(K/k)]$–submodule of U_{K,S_∞} generated by

$$\mu_K, \quad \mathcal{E}_{K,S_\infty,T},$$

for all sets T as above, such that $S_\infty \cap T = \emptyset$.

A direct application of Kolyvagin's Euler System techniques (see [**Ru1**], [**Ru2**] and [**Ru4**]) relying on the fundamental fact that each of the generators of $\mathcal{E}_{K,S_\infty,T}$ which is not contained in the base field k is the starting point of an Euler System of units (see §3.1) leads to a proof of parts (2) and (3) of the following result. The proof of part (1) is elementary (see [**P3**]).

Theorem 3.2.3. *If $\mathrm{char}(k) = 0$ and conjecture $B(L/k, S_L, T, r)$ is true for all $L \in \mathcal{K}$, then*

1. *The indices $[U_{K,S_\infty,T} : \mathcal{E}_{K,S_\infty,T}]$ and $[U_{K,S_\infty} : \mathcal{E}_{K,S_\infty}]$ are finite.*
2. *For all ℓ prime, $\ell \nmid \mathrm{card}(G(K/k))$, and all $\chi \in \widehat{G(K/k)}$, we have*
$$\mathrm{card}((A_{K,S_\infty,T} \otimes \mathbb{Z}_\ell)^\chi)^r = \mathrm{card}((U_{K,S_\infty,T}/\mathcal{E}_{K,S_\infty,T} \otimes \mathbb{Z}_\ell)^\chi).$$
3. *For all ℓ prime, $\ell \nmid \mathrm{card}(G(K/k)) \cdot \mathrm{card}(\mu_K)$, and all $\chi \in \widehat{G(K/k)}$, we have*
$$\mathrm{card}((A_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi)^r = \mathrm{card}((U_{K,S_\infty}/\mathcal{E}_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi).$$

In the case $\mathrm{char}(k) = p$, one could try to prove the theorem above by employing the Euler System technique as well. Such an attempt would be successful with a single, but extremely important exception. In (2) and (3) one would be forced to assume that $\ell \neq p$, because in characteristic p the Euler System technique fails to give upper bounds for the p–primary part of the Selmer group in question (which in the present setting is an ideal class–group). However, in [**P3**] we construct the groups of special units $\mathcal{E}_{K,S_\infty,T}$ and $\mathcal{E}_{K,S_\infty}$ in characteristic p and prove the analogue of Theorem 4.2.3 above **unconditionally** and without making use of Kolyvagin's Euler System technique. In fact, we show that if conjecture $\mathbb{Z}[1/|G(K'/k)|]B(K'/k, S', T', r')$ holds true, for all $K' \subseteq K$ and all S', T', r', such that $(K'/k, S', T', r')$ satisfies $(H_{r'})$, then the analogue of Theorem 4.2.3 follows. As conjecture $\mathbb{Z}[1/|G(K'/k)|]B(K'/k, S', T', r')$ was proved in full generality in characteristic p in [**P2**] (see §5 ???), we have the following (see [**P3**]).

THEOREM 3.2.4. *If* $\mathrm{char}(k) = p$*, then*

1. *The indices* $[U_{K,S_\infty,T} : \mathcal{E}_{K,S_\infty,T}]$ *and* $[U_{K,S_\infty} : \mathcal{E}_{K,S_\infty}]$ *are finite.*
2. *For all* ℓ *prime,* $\ell \nmid \mathrm{card}(G(K/k))$*, and all* $\chi \in \widehat{G(K/k)}$*, we have*

$$\mathrm{card}((A_{K,S_\infty,T} \otimes \mathbb{Z}_\ell)^\chi)^r = \mathrm{card}((U_{K,S_\infty,T}/\mathcal{E}_{K,S_\infty,T} \otimes \mathbb{Z}_\ell)^\chi)\,.$$

3. *For all* ℓ *prime,* $\ell \nmid \mathrm{card}(G(K/k)) \cdot \mathrm{card}(\mu_K)$*, and all* $\chi \in \widehat{G(K/k)}$*, we have*

$$\mathrm{card}((A_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi)^r = \mathrm{card}((U_{K,S_\infty}/\mathcal{E}_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi)\,.$$

In the particular case when $r = 1$ (i.e. when S_∞ consists of a single prime), we manage to improve (3) above slightly by proving the following (see [**P3**]).

THEOREM 3.2.5. *If* $\mathrm{char}(k) = p$ *and* $r := \mathrm{card}(S_\infty) = 1$*, then*

$$\mathrm{card}((A_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi) = \mathrm{card}((U_{K,S_\infty}/\mathcal{E}_{K,S_\infty} \otimes \mathbb{Z}_\ell)^\chi)\,,$$

for all prime numbers ℓ*, such that* $\ell \nmid \mathrm{card}(G(K/k))$*.*

Theorem 3.2.5 plays a crucial role in our proof of Chinburg's Ω_3–conjecture for prime degree Galois extensions of function fields (see [**P3**]), a result also obtained by Bae earlier and with different methods (see [**Bae**]).

EXAMPLES AND REMARKS. In the case where $k = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_m)^+$ (for which $S_\infty = \{\infty\}$, and therefore $r = 1$), the group of Stark units $\mathcal{E}_{K,S_\infty}$ constructed above contains the classical group of cyclotomic units $\mathcal{C}_m$, constructed by Sinnott in [**Si**], with a finite, possibly non–trivial 2–power index. The reader can prove this statement without difficulty, as a consequence of the Remark at the end of §3.1. Therefore, if restricted to this case, Theorem 3.2.3(3) above leads to the classical Gras Conjecture.

THEOREM (the Gras Conjecture). *Let* ℓ *be an odd prime number with the property that* $\ell \nmid \mathrm{card}(G(\mathbb{Q}(\zeta_m)^+/\mathbb{Q}))$*. Then, for all* $\chi \in \widehat{G(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})}$*, we have*

$$\mathrm{card}((A_m \otimes \mathbb{Z}_\ell)^\chi) = \mathrm{card}((U_m/\mathcal{C}_m \otimes \mathbb{Z}_\ell)^\chi)\,,$$

where A_m *and* U_m *are the ideal–class group and the group of global units of* $\mathbb{Q}(\zeta_m)^+$*.*

The theorem above was proved first by Mazur and Wiles in [**MW**], as a consequence of their proof of the Iwasawa Main Conjecture over $\mathbb{Q}$, and following Greenberg's earlier insightful remarks [**Gr1**], [**Gr2**]. Later, Kolyvagin and Rubin (see for example [**Ru2**]) gave a different proof, based on Kolyvagin's Euler System techniques applied to the particular case of Euler Systems of cyclotomic units.

The groups of special units $\mathcal{E}_{K,S_\infty}$ constructed above should be viewed as generalizations of Sinnott's groups of cyclotomic units $\mathcal{C}_K$ (if $k = \mathbb{Q}$), elliptic units (if k is a quadratic imaginary field), or Hayes elliptic units [**H4**] (if k is a function field). Theorems 3.2.3, 3.2.4, and 3.2.5 should be regarded as generalizations of the classical Gras Conjecture stated above.

4. Gross's refinement of the Rubin–Stark conjecture

With notations as in §1 above, we fix $r \in \mathbb{Z}_{\geq 0}$ and assume that the set of data $(K/k, S, T, r)$ satisfies hypotheses (H_r). In the case $r = 1$, Gross [**Gro1–2**] proposed a refinement of the Rubin–Stark conjecture $B(K/k, S, T, r)$. If the distinguished split prime is finite, Gross's conjecture predicts the $\wp$–adic expansion of the S–unit $\varepsilon_{S,T} \in U_{S,T}$ at a prime $\wp$ in S_K sitting above the split prime in terms of values of derivatives of p–adic L–functions. In the function field case, this statement was proved by Hayes in [**H2**]. In the case where $k = \mathbb{Q}$ and K imaginary this statement was proved by Gross in [**Gro1**] (see also [**GrKo**] and [**FG**].) In the early 1990s, Gross and Tate [**Gro3**] expressed interest in formulating a Gross–type refinement for the Rubin–Stark conjecture for arbitrary orders of vanishing r. Tan formulated and partly proved such a refinement for function fields (see [**T**].) In this section, we describe a general Gross–type refinement of the Rubin-Stark Conjecture and, in the number field case, interpret it in terms of special values of derivatives of p–adic L–functions.

4.1. Evaluation maps. The main point behind the conjecture we are about to describe is a reinterpretation of Rubin's lattice in terms of evaluation maps taking values in group rings with graded rings of coefficients. We remind the reader that Rubin's lattice in this context is defined by

$$\Lambda_{S,T} = \left\{ \varepsilon \in (\mathbb{Q}\overset{r}{\wedge} U_{S,T})_{r,S} \;\middle|\; \begin{array}{l} (\phi_1 \wedge \ldots \wedge \phi_r)(\varepsilon) \in \mathbb{Z}\,[G] \\ \forall\, \phi_1, \ldots, \phi_r \in U_{S,T}^* := \operatorname{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}\,[G]) \end{array} \right\},$$

where $(\phi_1 \wedge \ldots \wedge \phi_r)(\varepsilon_1 \wedge \ldots \wedge \varepsilon_r) := \det(\phi_i(\varepsilon_j))$, for all $\varepsilon_1, \ldots, \varepsilon_r \in U_{S,T}$. This shows that every $\varepsilon \in \Lambda_{S,T}$ gives rise to a $\mathbb{Z}\,[G]$–equivariant evaluation map

$$\mathrm{ev}_{\varepsilon,\mathbb{Z}} : \overset{r}{\wedge} U_{S,T}^* \longrightarrow \mathbb{Z}\,[G]\,, \qquad \mathrm{ev}_{\varepsilon,\mathbb{Z}}(\phi_1 \wedge \ldots \wedge \phi_r) := (\phi_1 \wedge \ldots \wedge \phi_r)(\varepsilon)\,.$$

However, it turns out that any fixed element $\varepsilon \in \Lambda_{S,T}$ gives rise to much more general evaluation maps. Indeed, let R be any commutative ring with 1. It is easy to see that we have a canonical $R\,[G]$–isomorphism

$$(U_{S,T})_R^* := \operatorname{Hom}_{\mathbb{Z}[G]}(U_{S,T}, R\,[G]) \xrightarrow{\sim} U_{S,T}^* \otimes R\,.$$

It turns out that the isomorphism above leads to a canonical evaluation map with values in $R[G] \simeq \mathbb{Z}[G] \otimes R$

$$\mathrm{ev}_{\varepsilon,R} : \overset{r}{\wedge} (U_{S,T})_R^* \longrightarrow \mathbb{Z}\,[G] \otimes R\,,$$

such that, for all $\phi_1, \ldots, \phi_r \in U_{S,T}^*$ and all $a_1, \ldots, a_r \in R$, we have

$$\mathrm{ev}_{\varepsilon,R}(\phi_1 \otimes a_1 \wedge \ldots \wedge \phi_r \otimes a_r) := \mathrm{ev}_{\varepsilon,\mathbb{Z}}(\phi_1 \wedge \ldots \wedge \phi_r) \otimes \prod a_i\,.$$

Remark. It is easy to see that if $R = \bigoplus_{i \geq 0} R^{(i)}$ is a graded ring and we pick elements $\psi_1, \ldots, \psi_r \in \operatorname{Hom}(U_{S,T}, R^{(1)}[G])$, then

$$\mathrm{ev}_{\varepsilon,R}(\psi_1 \wedge \ldots \wedge \psi_r) \in \mathbb{Z}[G] \otimes R^{(r)}\,.$$

4.2. The relevant graded rings R. In what follows, L/k will be an abelian, *not necessarily finite* extension of k containing K, such that the data $(L/k, S, T)$ satisfies hypotheses (H_0). We let $\Gamma := \mathrm{Gal}(L/K)$ and $\mathcal{H} := \mathrm{Gal}(L/k)$. We let $I(\Gamma) \subset \mathbb{Z}[[\Gamma]]$ and $I_\Gamma \subseteq \mathbb{Z}[[\mathcal{H}]]$ denote the usual augmentation and relative augmentation ideal associated to Γ inside the appropriate profinite group algebras. If L/k is not finite, then we let

$$\Theta_{L/k,S,T}(0) := \varprojlim_{L'/k} \Theta_{L'/k,S,T}(0) \in \mathbb{Z}[[\mathcal{H}]]\,,$$

where the projective limit is taken with respect to all finite extensions L'/k, with $k \subseteq L' \subseteq L$ and the Galois-restriction maps and the level of the corresponding (finite) group–rings.

Definition. We define R_Γ to be the graded ring

$$R_\Gamma := \bigoplus_{n\geq 0} I(\Gamma)^n / I(\Gamma)^{n+1}\,,$$

with the obvious addition and multiplication.

For all n, we have the following isomorphisms of $\mathbb{Z}[G]$–modules

$$I(\Gamma)^n/I(\Gamma)^{n+1} \otimes \mathbb{Z}[G] \xrightarrow{\sim} I_\Gamma^n/I_\Gamma^{n+1}$$
$$\widehat{x} \otimes \sigma \longrightarrow \widehat{x\cdot\widetilde{\sigma}}\,,$$

where $\widehat{x}$ is the class of $x \in I(\Gamma)^n$ in the quotient $I(\Gamma)^n/I(\Gamma)^{n+1}$ and $\widetilde{\sigma}$ is an arbitrary lift of $\sigma \in G$ to Γ (with respect to the natural restriction map $\Gamma \twoheadrightarrow G$.) Obviously, we obtain the following isomorphisms of $\mathbb{Z}[G]$–modules.

$$\bigoplus_{n\geq 0} I_\Gamma^n/I_\Gamma^{n+1} \xrightarrow{\sim} R_\Gamma \otimes \mathbb{Z}[G] \xrightarrow{\sim} R_\Gamma[G]\,.$$

4.3. The Conjecture. As above, we assume that $(K/k, S, T, r)$ satisfy (H_r) and L/k is an abelian extension (not necessarily finite), such that $k \subseteq K \subseteq L$ and $(L/k, S, T)$ satisfy (H_0). As in the statement of Rubin's conjecture for $(K/k, S, T, r)$, we let $\{v_1, \dots, v_r\}$ be an ordered set of r distinct primes in S which split completely in K/k and for each $i = 1, \dots, r$, we fix a prime w_i in K sitting above v_i. Let $v := v_i$ and $w := w_i$, for some i. Let

$$\rho_w : K_w^\times \longrightarrow G_w(L/K) \subseteq \Gamma$$

be the local Artin reciprocity map associated to w. Following Gross, we define the following $\mathbb{Z}[G]$–equivariant maps, for all w as above.

$$\phi_w : U_{S,T} \longrightarrow \mathbb{Z}[G] \otimes R_\Gamma^{(1)}\,, \quad \phi_w(u) = \sum_{\sigma\in G} \sigma^{-1} \otimes (\rho_w\widehat{(u^\sigma)} - 1)\,,$$

where $(\rho_w\widehat{(u^\sigma)} - 1)$ is the class of $(\rho_w(u^\sigma) - 1)$ in $I(\Gamma)/I(\Gamma)^2 = R_\Gamma^{(1)}$.

Conjecture $B(L/K/k, S, T, r)$. *Assume that $(L/K/k, S, T, r)$ satisfy all the above hypotheses. Then the following hold.*

1. $\Theta_{L/k,S,T}(0) \in I_\Gamma^r$.
2. *Assume that the Rubin-Stark Conjecture $B(K/k, S, T, r)$ holds. Let $\varepsilon := \varepsilon_{S,T}$ denote the (unique) Rubin–Stark element for data $(K/k, S, T, r)$. Then*

$$\Theta_{L/k,S,T}(0) \bmod I_\Gamma^{r+1} = \mathrm{ev}_{\varepsilon, R_\Gamma}(\phi_{w_1} \wedge \ldots \wedge \phi_{w_r}) \text{ in } I_\Gamma^r / I_\Gamma^{r+1}.$$

The following proposition makes the link between the statement above and the classical global conjectures and the p–adic conjecture of Gross (Conjectures 4.1 and 7.6 in [**Gro2**] and Conjecture 3.13 in [**Gro1**], respectively.)

Proposition. *Assume that $(L/K/k, S, T, r)$ satisfy the above hypotheses. The following hold true.*

1. *If $K = k$, then*

$$B(L/K/k, S, T, r) \iff \text{Gross's Global Conjecture 4.1 [Gro2]}.$$

2. *If $r = 1$, then*

$$B(L/K/k, S, T, r) \iff \text{Gross's Global Conjecture 7.1 [Gro2]}.$$

3. *Assume that K is CM, k is totally real and $L := K_{p^\infty}$ is the cyclotomic $\mathbb{Z}_p$–extension of K, for some prime number p. Then*

$$B(L/K/k, S, T, 1) \iff \text{Gross's } p\text{–adic Conjecture 3.13 [Gro1]}.$$

4.4. Linking values of derivatives of p–adic and global L–functions. One of the main features of the Conjecture stated in §4.3 above is that, under the appropriate hypotheses, it establishes a deep connection between the values of the r–th derivatives of (equivariant) global and p–adic L–functions at $s = 0$. We describe this connection below.

Let us assume that $\mathrm{char}(k) = 0$, p is a prime number and $L := K_{p^\infty}$ is the cyclotomic $\mathbb{Z}_p$–extension of K. Note that under these assumptions we have $S_p \subseteq S$, where S_p denotes the set of all primes in k sitting above p. In this case, we have isomorphisms

$$\Gamma \xrightarrow{\sim} \mathbb{Z}_p, \qquad I(\Gamma)^n / I(\Gamma)^{n+1} \xrightarrow{\sim} \mathbb{Z}_p, \qquad I_\Gamma^n / I_\Gamma^{n+1} \xrightarrow{\sim} \mathbb{Z}_p[G],$$

for all $n \geq 1$. (Note that the first two maps above are group isomorphisms, while the third is a $\mathbb{Z}_p[G]$–module isomorphism.) For simplicity, let us assume further that $e^{2\pi i/p} \in K$ and that k_{p^∞} and K are linearly disjoint over k. Consequently, we have a group isomorphism

$$\mathcal{H} \xrightarrow{\sim} \Gamma \times G.$$

Also, after picking a generator γ of Γ, we have a canonical ring isomorphism

$$\mathbb{Z}_p[[\mathcal{H}]] \xrightarrow{\sim} \mathbb{Z}_p\,[G]\,[[t]],$$

which sends $\gamma - 1$ to t. Via this isomorphism, the equivariant L–value $\Theta_{L/k,S,T}(0)$ can be viewed as a power series of variable t with coefficients in the group ring $\mathbb{Z}_p[G]$. Also, it is easily seen that via this isomorphism we have

$$I_\Gamma^n \xrightarrow{\sim} t^n \mathbb{Z}_p[G][[t]]\,,$$

for all n. Consequently, if $\Theta_{L/k,S,T}(0) \in I_\Gamma^r$, we have

$$\Theta_{L/k,S,T}(0) \mod I_\Gamma^{r+1} = \frac{1}{r!}\cdot \frac{d^r}{dt^r}(\Theta_{L/k,S,T}(0))\mid_{t=0} \in \mathbb{Z}_p[G]\,.$$

Now, let us assume that conjecture $B(K/k,S,T,r)$ and $B(L/K/k,S,T,r)$ hold true. Let $\varepsilon := \varepsilon_{S,T}$ denote the Rubin-Stark element. The previous observations show that conjecture $B(L/K/k,S,T,r)$ is equivalent to

$$\Theta_{L/k,S,T}(0) \in t^r \mathbb{Z}_p[G][[t]]\,, \quad \frac{1}{r!}\frac{d^r}{dt^r}(\Theta_{L/k,S,T}(0))\mid_{t=0} = \mathcal{R}_p(\varepsilon)\,,$$

where

$$\mathcal{R}_p : \Lambda_{S,T} \longrightarrow \mathbb{Z}_p[G]\,, \quad \mathcal{R}_p(\eta) := \mathrm{ev}_{\eta,R_\Gamma}(\phi_{w_1}\wedge \cdots \wedge \phi_{w_r})$$

is Gross's (p-adic) regulator defined in §4.3 above for all $\eta \in \Lambda_{S,T}$.

Now, the main point is that $\Theta_{L/k,S,T}(0) \in \mathbb{Z}_p[G][[t]]$ is the G–equivariant p–adic L–function associated to the set of data $(K/k,S,T,r)$. We clarify this connection (essentially due to Iwasawa) below.

For this purpose, we let

$$\omega_p, \chi_p : \mathrm{Gal}(L/k) \longrightarrow \mathbb{Z}_p^\times$$

denote the p–adic Teichmüller and cyclotomic characters of $\mathrm{Gal}(L/k)$, respectively. Note that under our assumptions ($e^{2\pi i/p} \in K$), the Teichmüler character ω_p factors through $G := \mathrm{Gal}(K/k)$. Let $u := \chi_p\omega_p^{-1}(\gamma) \in 1 + p\mathbb{Z}_p$. Due to Iwasawa, we know that

$$\Theta_{L/k,S,T}(0) = \sum_{\chi\in\widehat{G}(\mathbb{C}_p)} f_\chi(t)\cdot e_\chi\,,$$

where $f_\chi \in \mathbb{Z}_p(\chi)[[t]]$ are power series uniquely determined by the remarkable interpolation property

$$f_\chi(u^{1-n}-1) = L_{S,T}(\chi^{-1}\omega_p^{1-n}, 1-n)\,,$$

for all $\chi \in \widehat{G}(\mathbb{C}_p)$ and all $n \in \mathbb{Z}_{\geq 1}$.

Definition. For $\chi \in \widehat{G}(\mathbb{C}_p)$, the (S,T)–modified p–adic L–function $L_p(\chi, s)$ of variable $s \in \mathbb{Z}_p$ is defined by

$$L_p(\chi, s) := f_{\chi^{-1}\omega_p}(u^s - 1)\,.$$

The definition above explains why we are entitled to think of $\Theta_{L/k,S,T}(0)$ as the G–equivariant (S,T)–modified p–adic L–function associated to the $(K/k,S,T)$. The following Proposition follows immediately from the above considerations.

PROPOSITION. *Let $(L/K/k,S,T,r)$ as above. Then conjectures $B(K/k,S,T,r)$ and $B(L/K/k,S,T,r)$ are true if and only if both statements below hold true.*

1. $ord_{s=0}L_p(\chi,s)\geq r$*, for all* $\chi\in\widehat{G}(\mathbb{C}_p)$.
2. *There exists a unique element* $\varepsilon:=\varepsilon_{S,T}\in\Lambda_{S,T}$*, such that*

$$\chi(\mathcal{R}_{gl}(\varepsilon))=\tfrac{1}{r!}\cdot\tfrac{d^r}{ds^r}L^{(r)}_{K/k,S,T}(\chi^{-1},s)\mid_{s=0}$$
$$\chi^{-1}\omega_p(\mathcal{R}_p(\varepsilon))=\tfrac{1}{(\log_p u)^r}\cdot\tfrac{1}{r!}\cdot\tfrac{d^r}{ds^r}L_p(\chi,s)\mid_{s=0},$$

where $\mathcal{R}_{gl}$ and $\mathcal{R}_p$ are the (global) Rubin and (p–adic) Gross regulators, respectively.

Conclusion. We conclude by remarking that, under the above hypotheses, the conjectures of Rubin-Stark and Gross imply the existence of a unique *global*, arithmetically meaningful element ε, whose evaluation against the global regulator $\mathcal{R}_{gl}$ essentially equals the r–th derivative of the G–equivariant global L–function at $s=0$ and whose evaluation against the p–adic regulator $\mathcal{R}_p$ essentially equals the r–th derivative of the G–equivariant p–adic L–function at $s=0$. Another remarkable consequence of these conjectures is that if the G–equivariant global L–function has order of vanishing at least r at $s=0$ due to the presence of r distinct primes in S which split completely in K/k, then the G–equivariant p–adic L-function has the same property.

5. Evidence in support of the Rubin–Stark and Gross Conjectures

TO BE WRITTEN

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