Special values of L-functions at negative integers

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Contents

0 Introduction

Let χ be a 1-dimensional Artin character over a number field F. The values of the Artin L-function $L_F(\chi, s)$ at negative integers $1-n$ for $n \geq 2$ are trivial unless F is totally real and χ acts on complex conjugation as $(-1)^n$, i.e. the field $F_\chi := \bar{F}^{ker\,\chi}$ is totally real for even n and a CM-field for odd n. In these cases the values are non-zero algebraic numbers contained in $\mathbb{Q}(\chi)$, the field obtained by adjoining to $\mathbb{Q}$ the values of χ.

In the lectures we will discuss the arithmetic meaning of these values. The approach is via Iwasawa theory, p-adic L-functions and the Main Conjecture in Iwasawa theory (proved by Wiles), which provides a p-adic interpretation of the values for each prime p. In the case of the trivial character we will describe the relation to the Birch-Tate Conjecture (the case where F is real and $n = 2$) as well as to the more general Lichtenbaum Conjectures. For most of the part we will ignore the prime 2, which causes technical problems, and yields less complete results.

The arithmetic interpretations for a fixed prime p are in terms of étale cohomology groups attached to the ring $o'_F = o_F[1/p]$ of p-integers of F. We will discuss two "global" interpretations in terms of algebraic K-groups and in terms of motivic cohomology groups, which may differ by powers of 2. The known results for $p = 2$ suggest that in general motivic cohomology contains the "correct" number-theoretic information.

Finally, we will discuss (in the "semi-simple" situation) a conjecture of Coates-Sinnott – the analog of Stickelberger's Theorem – about annihilation of higher algebraic K-theory groups in a relative abelian extension.

These notes contain more background information than could possibly be covered in the lectures, and many references for further reading. The aim is to present the current state of the art in this field, and introduce some of the key techniques involved.

1 The Classical Main Conjecture

Let F be a number field and let p be a prime number. A Galois extension F_∞/F is called a $\mathbb{Z}_p$-*extension*, if $\Gamma := Gal(F_\infty/F) \cong \mathbb{Z}_p$. Since the closed subgroups of $\mathbb{Z}_p$ are of the form 0 or $p^n\mathbb{Z}_p$, we have for each $n \geq 0$ a unique subfield F_n of degree p^n over F and $Gal(F_n/F) \cong \mathbb{Z}/p^n\mathbb{Z}_p$. Hence we obtain a tower

$$F = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\infty,$$

such that $[F_n : F] = p^n$ and $F_\infty = \cup_{n\geq 0} F_n$.

A typical example of a $\mathbb{Z}_p$-extension F_∞/F is the so-called *cyclotomic* $\mathbb{Z}_p$-*extension*, which is constructed as follows: Let $L_\infty = F(\mu_{p^\infty})$. Then $Gal(L_\infty/F) \cong \mathbb{Z}_p \times \Delta$, where Δ is finite. Now take $F_\infty = L_\infty^\Delta$.

Let γ denote a topological generator of Γ, and let $\Gamma_n = Gal(F_n/F)$. Passing to the inverse limit over the group rings $\mathbb{Z}_p[\Gamma_n]$ we obtain the *Iwasawa-algebra* $\mathbb{Z}_p[[\Gamma]] := \varprojlim \mathbb{Z}_p[\Gamma_n]$. The group rings $\mathbb{Z}_p[\Gamma_n]$ are generally quite complicated, but the Iwasawa-algebra has a rather simple structure, it is isomorphic to the power series ring $\Lambda := \mathbb{Z}_p[[T]]$, the isomorphism being induced by $\gamma \mapsto 1 + T$.

In the following we have to allow slightly more general coefficients: Let $\mathcal{O}$ denote a finite extension of $\mathbb{Z}_p$, let π be a uniformizer for $\mathcal{O}$, let v denote the discrete valuation on $\mathcal{O}$, normalized so that $v(\pi) = 1$, and let $|\ \ |_v$ denote the corresponding absolute value with $|a|_v = p^{-f\cdot v(a)}$, where f denotes the residue degree.

We now consider $\Lambda := \mathcal{O}[[T]] \cong \mathcal{O}[[\Gamma]]$. This is a two-dimensional Noetherian local Krull domain, and the structure of finitely generated Λ-modules is known up to pseudo-isomorphism. If M and N are finitely generated Λ-modules, then we write $M \sim N$ if there exists a pseudo-isomorphism $f : M \to N$, *i.e.*, a module homomorphism with finite kernel and cokernel. The structure theorem for finitely generated Λ-modules now says that for every finitely generated Λ-module M there is a pseudo-isomorphism

$$M \sim \Lambda^r \oplus \bigoplus_{i=1}^{m} \Lambda/\mathfrak{p}_i^{n_i}.$$

Here $\mathfrak{p}_i$ are height 1 prime ideals of Λ, hence they are either equal to (π) or to $(F(T))$, where $F(T)$ is an irreducible Weierstrass polynomial, *i.e.*, of the form

$$F(T) = T^n + b_{n-1}T^{n-1} + \cdots + b_0$$

with $\pi | b_i$ for all i. The prime ideals $\mathfrak{p}_i$ and the integers $r \geq 0, m \geq 0$ and $n_i \geq 1$ are uniquely determined by M. The ideal $\prod_{i=1}^{m} \mathfrak{p}_i^{n_i}$ is the *characteristic ideal* of M, which has a unique generator of the form

$$f(T) = \pi^{\mu} \cdot f^*(T)$$

where $f^*(T)$ is a Weierstrass polynomial. $f^*(T)$ is the *characteristic polynomial* of M. The exponent μ is the *μ-invariant of M* and $\lambda := deg\, f^*(T)$ is called the *λ-invariant of M*.

The characteristic polynomial is in fact a characteristic polynomial in the sense of linear algebra: Let $\overline{\mathbb{Q}_p}$ denote an algebraic closure of $\mathbb{Q}_p$, and let $V = M \otimes_{\mathcal{O}} \overline{\mathbb{Q}_p}$. This is a $\overline{\mathbb{Q}_p}$-vectorspace of rank λ and $f^*(T)$ is the characteristic polynomial of the endomorphism $\gamma - 1$ acting on V.

The following result is extremely useful: Assume that M is a finitely generated Λ-torsion module with characteristic polynomial $f^*(T)$. Let μ denote the μ-invariant of M and let $f(T) = \pi^{\mu} \cdot f^*(T)$. We denote by M^{Γ} the invariants of M under Γ and by $M_{\Gamma} = M/(\gamma - 1)M$ the coinvariants of M.

Lemma 1.1 ((cf. [26]). *The following statements are equivalent :*

(a) M^{Γ} is finite
(b) M_{Γ} is finite
(c) $f(0) \neq 0$.

If these conditions are satisfied, then

$$\frac{|M^{\Gamma}|}{|M_{\Gamma}|} = |f(0)|_v.$$

Let us assume now that F_{∞}/F is the cyclotomic $\mathbb{Z}_p$-extension. Let $L_{\infty} = F(\mu_{p^{\infty}})$ and let $G_{\infty} = Gal(L_{\infty}/F) \cong \Gamma \times \Delta$, where $\Delta \cong Gal(F(\zeta_{2p})/F$. Since L_{∞} contains all p-power roots of unity, the Galois group G_{∞} acts on $\mu_{p^{\infty}}$ and this action gives rise to the *cyclotomic character*

$$\rho : G_{\infty} \to \mathbb{Z}_p^*$$

defined by

$$\zeta^{\sigma} = \zeta^{\rho(\sigma)}$$

for all $\sigma \in G_{\infty}$ and all $\zeta \in \mu_{p^{\infty}}$. We denote by κ the restriction of ρ to Γ and by ω the restriction of ρ to Δ. ω is the *Teichmüller character.*

Let M be a $\mathbb{Z}_p$-module with a G_∞-action, denoted by $m \mapsto m^\sigma$. For $n \in \mathbb{Z}$ the n-th Tate twist $M(n)$ of M is defined as the $\mathbb{Z}_p$-module M with the new G_∞-action

$$m \mapsto \rho(\sigma)^n \cdot m^\sigma.$$

In particular, $\mathbb{Z}_p(1) \cong \varprojlim \mu_{p^n} =: \mathcal{T}$, which is the so-called *Tate-module*, and $\mathbb{Q}_p/\mathbb{Z}_p(1) \cong \mu_{p^\infty}$. In general: $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$. If M and N are two $\mathbb{Z}_p$-modules with a G_∞-action, then we turn $Hom_{\mathbb{Z}_p}(M,N)$ into a G_∞-module in the following way: For $f \in Hom_{\mathbb{Z}_p}(M,N)$ and $\sigma \in G_\infty$ we define f^σ via

$$f^\sigma(m) = (f(m^{\sigma^{-1}}))^\sigma.$$

It is easy to see that with this definition of the G_∞-action on *Hom*-groups we obtain canonical isomorphisms for all $n \in \mathbb{Z}$:

$$Hom_{\mathbb{Z}_p}(M(n), \mathbb{Q}_p/\mathbb{Z}_p) \cong Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(-n)) \cong Hom_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)(-n).$$

We note the following:

Lemma 1.2 (cf. [26]). *Assume that M is a Λ-torsion module with characteristic polynomial $f^*(T)$. Then the characteristic polynomial of $M(n)$ is given by*

$$f^*(\kappa(\gamma)^{-n}(1+T)-1).$$

The most interesting Λ-modules arise as Galois groups of certain abelian pro-p extensions of F_∞, where F_∞/F is an arbitrary $\mathbb{Z}_p$-extension of a number field F. Assume then that K_∞ is an abelian pro-p extension of F_∞, let $X = Gal(K_\infty/F_\infty)$, and assume that K_∞/F is again a Galois extension (although not necessarily abelian). Let $G = Gal(K_\infty/F)$. We obtain an extension of $\mathbb{Z}_p$-modules

$$0 \to X \to G \to \Gamma \to 0.$$

Since X is abelian, Γ acts on X by inner automorphisms, and this action turns X into a compact Λ-module. As examples we can take for K_∞ the maximal abelian unramified pro-p extension of F_∞, usually denoted by L_∞, or the maximal subextension of L_∞, in which all p-adic primes of F_∞ split completely, usually denoted by L'_∞. The corresponding Galois groups $X_\infty := Gal(L_\infty/F_\infty)$ and $X'_\infty := Gal(L'_\infty/F_\infty)$ are examples of finitely generated Λ - torsion modules.

The main example in the current framework is the following: Let S be a finite set of primes in F containing the primes above p and the infinite primes. S_p will denote the minimal such set, i.e. the set consisting exactly of the primes above p and the infinite primes. Let M_∞^S denote the maximal abelian pro-p-extension of F_∞, which is unramified outside primes in S, and let $\mathfrak{X}^S = Gal\,(M_\infty^S/F_\infty)$. This is a finitely generated Λ-module, which we will call the *standard Iwasawa module* over F_∞ for the set S. Let us again specialize to the case of the cyclotomic $\mathbb{Z}_p$-extension. Iwasawa has shown that in this case $\mathfrak{X}^S$ has no non-trivial finite Λ-submodules and that the Λ-rank of $\mathfrak{X}^S$ is equal to the number r_2 of different pairs of complex conjugate embeddings of F. In particular, $\mathfrak{X}^S$ is a Λ-torsion module if and only if F is totally real.

From now on F will be a totally real number field and F_∞ will denote the cyclotomic $\mathbb{Z}_p$-extension with p being an odd prime number. As before, we let $L = F(\zeta_p)$ and $L_\infty = F(\mu_{p^\infty})$.

We consider now a 1-dimensional p-adic valued Artin character ψ over F of finite order:

$$\psi : Gal(\overline{\mathbb{Q}_p}/F) \to \overline{\mathbb{Q}_p}^*,$$

and we denote by F_ψ the fixed field of the kernel of ψ, so that ψ is a faithful character on $Gal(F_\psi/F)$. We assume that ψ is even, i.e. that F_ψ is again a totally real number field. We also recall Greenberg's terminology (cf. [16]) about the different *types* of the characters ψ: ψ is of *type* S, if

$$F_\psi \cap F_\infty = F,$$

and ψ is of *type* W, if

$$F_\psi \subset F_\infty.$$

We note that the only character of both types is the trivial character.

Deligne-Ribet ([13]) have shown that there exists a p-adic L-function $L_p(s,\psi)$, which interpolates the special values of Artin L-functions in the following way: For all $n \geq 1$

$$L_p(1-n,\psi) = L(1-n,\psi\omega^{-n}) \cdot \prod_{\mathfrak{p}|p} (1 - \psi\omega^{-n}(\mathfrak{p})N(\mathfrak{p})^{n-1}).$$

These values determine the p-adic L-function uniquely.

Now let S be again a finite set of primes in F containing S_p. Removing Euler factors at primes in $S \setminus S_p$ from $L_p(s, \psi)$ one also obtains an "imprimitive" p-adic L-function $L_p^S(s, \psi)$ with the property that for all $n \geq 1$

$$L_p^S(1-n, \psi) = L(1-n, \psi\omega^{-n}) \cdot \prod_{\mathfrak{p} \in S \setminus S_p} (1 - \psi\omega^{-n}(\mathfrak{p}) N(\mathfrak{p})^{n-1}).$$

The same truncation can be done to the Artin L-functions to obtain the L-functions L^S and the relation between the truncated L-functions at $1-n$ is then exactly the same as above (cf. [15]):

$$L_p^S(1-n, \psi) = L^S(1-n, \psi\omega^{-n}) \cdot \prod_{\mathfrak{p} | p} (1 - \psi\omega^{-n}(\mathfrak{p}) N(\mathfrak{p})^{n-1}).$$

We define

$$H_\psi(T) = \begin{cases} \psi(\gamma)(1+T) - 1 & \text{if } \psi \text{ is of type W} \\ 1 & \text{otherwise} \end{cases},$$

and we denote the extension of $\mathbb{Z}_p$ obtained by adjoining the values $\psi(g)$, $g \in Gal(F_\psi/F)$ by $\mathcal{O}_\psi = \mathbb{Z}_p[\psi]$. It was shown by Deligne-Ribet that there exists a power series $G_{\psi,S}(T) \in \mathcal{O}_\psi[[T]]$, so that

$$L_p^S(1-s, \psi) = \frac{G_{\psi,S}(\kappa(\gamma)^s - 1)}{H_\psi(\kappa(\gamma)^s - 1)}.$$

The power series $G_{\psi,S}(T)$ can be written uniquely as

$$G_{\psi,S}(T) = \pi^{\mu(G_{\psi,S})} \cdot g^*_{\psi,S}(T) \cdot u_{\psi,S}(T),$$

where $g^*_{\psi,S}(T)$ is a distinguished polynomial, $u_{\psi,S}(T)$ is a unit power series and π is a uniformizer in $\mathcal{O}_\psi$.

The classical Main Conjecture in Iwasawa Theory, proven by Wiles in [37] for odd p (and also for $p=2$, if $F = \mathbb{Q}$) relates the polynomial $g^*_{\psi,S}(T)$ to the following characteristic polynomial: Let $F_{\psi,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of F_ψ, and let $\mathfrak{X}^S$ denote the standard Iwasawa module over $F_{\psi,\infty}$ for the set S. The Galois group $G = Gal(F_\psi/F)$ acts on the finite-dimensional vectorspace

$$V = \mathfrak{X}^S \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p,$$

and we denote by V^ψ the eigenspace of V corresponding to the action of G via ψ. Now let $f^*_{\psi,S}(T)$ denote the characteristic polynomial of $\gamma - 1$ acting on V^ψ.

Iwasawa's Main Conjecture 1.3 (Wiles). *Let F be a totally real number field, let p be an odd prime, and let ψ be a 1-dimensional p-adic Artin character over F of type S. Then for any finite set of primes S of F containing S_p:*

$$g^*_{\psi,S}(T) = f^*_{\psi,S}(T).$$

It is important to note that the characteristic polynomial $f^*_{\psi,S}(T)$ does not change if we replace F_ψ by a finite extension E with E/F again abelian and $E \cap F_\infty = F$ and then consider the standard Iwasawa module over E_∞ instead. (cf. [16], Proposition 1).

2 Cohomology

We are going to use the Main Conjecture to relate special values of L-functions at negative integers to orders of étale cohomology groups. For our purposes it suffices to use a description of étale cohomology in terms of Galois cohomology: Fix an arbitrary prime p and an arbitrary number field F. Let $\Omega_F^{(p)}$ denote the maximal algebraic extension of F, which is unramified outside primes above p and infinite primes, and let $G_F^{(p)} = Gal(\Omega_F^{(p)}/F)$. The étale cohomology groups $H^*_{\text{ét}}(spec\, o_F[\frac{1}{p}], \mu_{p^m}^{\otimes n})$ of the scheme $spec\, o_F[\frac{1}{p}]$ with values in the étale sheaf $\mu_{p^m}^{\otimes n}$ as defined by Grothendieck (cf. e.g. [28]) can be identified with the Galois cohomology groups $H^*(G_F^{(p)}, \mu_{p^m}^{\otimes n})$. To simplify notations we will write $H^*_{\text{ét}}(o'_F, \mathbb{Z}/p^m(n))$, where $o'_F = o_F[\frac{1}{p}]$. Similarly, if S is a finite set of primes of F containing S_p, then we obtain the étale cohomology groups $H^*_{\text{ét}}(o_F^S, \mathbb{Z}/p^m(n))$ as Galois cohomology groups, where we replace the extension $\Omega_F^{(p)}$ by the maximal algebraic S-ramified extension Ω_F^S of F.

We will mainly be interested in the p-adic cohomology groups

$$H^*_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) := \varprojlim H^*_{\text{ét}}(o'_F, \mu_{p^m}^{\otimes n}),$$

which for $n \geq 2$ play a role, similar to the (p-parts of) class groups in classical class number formulas.

We also define

$$H^*_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim H^*_{\text{ét}}(o'_F, \mu_{p^m}^{\otimes n}).$$

We note the following: For each $n \in \mathbb{Z}$ the exact sequence

$$0 \to \mathbb{Z}_p(n) \to \mathbb{Q}_p(n) \to \mathbb{Q}_p(n)/\mathbb{Z}_p(n) \to 0$$

gives rise to a long exact sequence in étale cohomology and the kernels and cokernels of the boundary maps

$$\delta_i : H^{i-1}_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \to H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n)) \ \ (i \geq 1)$$

can be described as follows (cf. [34]): The kernel of δ_i is the maximal divisible subgroup of $H^{i-1}_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and the image of δ_i is the torsion subgroup of $H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$. In particular this implies that the torsion subgroup of $H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))$ is isomorphic to $H^0_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n))$:

$$H^1_{\text{ét}}(o'_F, \mathbb{Z}_p(n))_{\text{tors}} \cong H^0_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

In the following proposition we summarize some known results about the finitely generated p-adic étale cohomology groups for rings of integers. We only list the results for odd primes p and integers $n \geq 2$

Proposition 2.1. *Let p be an odd prime and let $n \geq 2$. Then*

1. $H^0_{ét}(o'_F, \mathbb{Z}_p(n)) = 0$.
2. $H^k_{ét}(o'_F, \mathbb{Z}_p(n)) = 0$ *for* $k \geq 3$.
3. *There are isomorphisms*

$$H^1_{ét}(o'_F, \mathbb{Z}_p(n)) \cong H^1_{ét}(F, \mathbb{Z}_p(n)).$$

4. *The groups $H^2_{ét}(o'_F, \mathbb{Z}_p(n))$ are finite and trivial for almost all primes p.*

5. $$rk_{\mathbb{Z}_p} H^1_{ét}(F, \mathbb{Z}_p(n)) = \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd } > 1 \\ r_2 & \text{if } n \text{ is even.} \end{cases}$$

One of the problems in dealing with the prime 2 is that property 2 is not true for $p = 2$ if F has real places.

Property 5. implies that for $n \geq 2$ the étale cohomology group $H^1_{\text{ét}}(F, \mathbb{Z}_p(n))$ is finite precisely, when F is totally real and n is even. If this is the case, then the boundary map δ_2 is an isomorphism:

$$H^1_{\text{ét}}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^2_{\text{ét}}(o'_F, \mathbb{Z}_p(n)).$$

Now let us assume that $n > 1$ is odd and that E is a CM-field with maximal real subfield E^+. Then $H^1_{\text{ét}}(E, \mathbb{Z}_p(n))$ has the same $\mathbb{Z}_p$-rank as $H^1_{\text{ét}}(E^+, \mathbb{Z}_p(n))$. Under complex conjugation $H^1_{\text{ét}}(E, \mathbb{Z}_p(n))$ splits into eigenspaces

$$H^1_{\text{ét}}(E, \mathbb{Z}_p(n)) = H^1_{\text{ét}}(E, \mathbb{Z}_p(n))^+ \oplus H^1_{\text{ét}}(E, \mathbb{Z}_p(n))^-$$

with $H^1_{\text{ét}}(E, \mathbb{Z}_p(n))^+ \cong H^1_{\text{ét}}(E^+, \mathbb{Z}_p(n))$. Therefore for odd $n > 1$ $H^1_{\text{ét}}(E, \mathbb{Z}_p(n))^-$ is finite and therefore

$$H^1_{\text{ét}}(o'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^- \cong H^2_{\text{ét}}(o'_E, \mathbb{Z}_p(n))^-.$$

We obtain:

Corollary 2.2. *a) If F is totally real and $n \geq 2$ is even, then*

$$H^1_{\acute{e}t}(o'_F, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^2_{\acute{e}t}(o'_F, \mathbb{Z}_p(n))$$

b) If E is CM, and $n > 1$ is odd, then

$$H^1_{\acute{e}t}(o'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))^- \cong H^2_{\acute{e}t}(o'_E, \mathbb{Z}_p(n))^-.$$

There are two global "cohomology theories", closely related to étale cohomology: Algebraic K-theory and motivic cohomology. The precise relationship depends on the validity of the Bloch-Kato Conjecture, which appears to have been proven by Rost and Voevodsky – at least all the details are now either published or submitted for publication. If we assume the Bloch-Kato Conjecture, then the picture is the following – the 2-primary information here is uncondtional:

For $i = 1, 2$ there are isomorphisms

$$K_{2n-i}(o_F) \cong H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n))$$

up to (known) 2-torsion, and for all p there are isomorphisms

$$H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n)) \otimes \mathbb{Z}_p \cong H^i_{\text{ét}}(o'_F, \mathbb{Z}_p(n)).$$

Here the K-groups are Quillen's K-groups, and the motivic cohomology groups can e.g. be defined as Bloch's higher Chow groups:

$$H^i_{\mathcal{M}}(o_F, \mathbb{Z}(n)) := CH^n(o_F, \mathbb{Z}(2n-i)).$$

If we do not want to assume the Bloch-Kato Conjecture, then we can still find global cohomological "models" $H^i(o_F, \mathbb{Z}(n))$, $i = 1, 2$, for the étale cohomology groups. For $i = 2$ this is easy. We simply define

$$H^2(o_F, \mathbb{Z}(n)) = \prod_p H^2_{\text{ét}}(o_F[\frac{1}{p}], \mathbb{Z}_p(n)).$$

For $i = 1$ the construction is more involved (cf. [8]).

In any case it is important to note that for certain indices there is a difference between the 2-primary parts of the K-groups and the cohomology groups, which has an impact on some of the conjectures we are going to discuss.

Assume now that E/F is a finite Galois extension of number fields with Galois group G. Let p be an odd prime and let S denote a finite set of primes of F containing all primes above p as well as all primes which ramify in F, so that the extension E/F is S-ramified. Using the properties of the étale cohomology groups the following results about Galois descent and co-descent follow from the Hochschild-Serre and the Tate spectral sequences:

Proposition 2.3. *Let E/F be a Galois extension of number fields with Galois group G. Let p be an odd prime and let S be a finite set of primes of F containing S_p and all primes ramified in E. Then*

1. $$H^1_{ét}(E, \mathbb{Z}_p(n))^G \cong H^1_{ét}(F, \mathbb{Z}_p(n))$$

2. $$H^2_{ét}(o_E^S, \mathbb{Z}_p(n))_G \cong H^2_{ét}(o_F^S, \mathbb{Z}_p(n)).$$

3. *For all $k \geq 0$ there are isomorphisms*

$$\hat{H}^k(G, H^1_{ét}(E, \mathbb{Z}_p(n))) \cong \hat{H}^k(G, H^2_{ét}(o_E^S, \mathbb{Z}_p(n))).$$

3 The Lichtenbaum Conjecture

We are now ready to apply the Main Conjecture in the case, where ψ is of order prime to p (p odd). The character ψ is then automatically of type S. We now choose a finite abelian extension E of F containing F_ψ and also μ_p, so that the Galois group G of E/F is of order prime to p. Then E_∞

contains all p-power roots of unity. For any finite set S of primes containing S_p the standard Iwasawa module $\mathfrak{X}^S$ over E_∞ is a $\mathbb{Z}_p[G][[T]]$-module. The following arguments do not depend on the choice of S, and so we simply drop the index S from the notations.

Since the order of G is prime to p, the idempotents of the group algebra $\mathbb{Q}_p[G]$ are contained in $\mathbb{Z}_p[G]$ and $\mathbb{Z}_p[G]$ is a maximal order in $\mathbb{Q}_p[G]$, isomorphic to a finite product of discrete valuation rings $\mathcal{O}_\rho$ for certain (absolutely irreducible) characters ρ of G. Given a finitely generated $\mathbb{Z}_p[G][[T]]-$ module M and a character ρ, the ρ-th component M^ρ of M is defined as

$$M^\rho = e_\rho(M \otimes_{\mathbb{Z}_p} \mathcal{O}_\rho).$$

This is a finitely generated $\mathcal{O}_\rho[[T]]-$ module.

We now take $M = \mathfrak{X}$ and $\rho = \psi$, and let $\Lambda = \mathcal{O}_\psi[[T]]$. Since ψ is even, the ψ-component $\mathfrak{X}^\psi$ is a finitely generated Λ-torsion module. We also note that $\mathcal{O}_\psi$ is unramified over $\mathbb{Z}_p$, and so we can take $\pi = p$ as the uniformizer. We denote the characteristic polynomial of $\mathfrak{X}^\psi$ by $f_\psi^*(T)$, and we let

$$f(T) = p^\mu \cdot f_\psi^*(T).$$

Wiles has shown ([37],Theorem 1.4) that the μ-invariant μ of $\mathfrak{X}^\psi$ coincides with $\mu(G_\psi)$, and therefore by the Main Conjecture the characteristic ideal of $\mathfrak{X}^\psi$ is generated by $G_\psi(T)$.

Let us fix now an integer $n \geq 2$ and consider the Λ-module $\mathfrak{X}^\psi(-n)$ and its Pontryagin dual $Hom_{\mathbb{Z}_p}(\mathfrak{X}^\psi(-n), \mathbb{Q}_p/\mathbb{Z}_p)$. We let $\chi = \psi\omega^{-n}$, and note that

$$\mathfrak{X}^\psi(-n) = \mathfrak{X}(-n)^\chi.$$

Taking duals we obtain

$$Hom_{\mathbb{Z}_p}(\mathfrak{X}^\psi(-n), \mathbb{Q}_p/\mathbb{Z}_p) = Hom_{\mathbb{Z}_p}(\mathfrak{X}^\chi, \mathbb{Q}_p/\mathbb{Z}_p(n)) = Hom_{\mathbb{Z}_p}(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\chi^{-1}}.$$

Because E_∞ contains all p-power roots of unity, the Galois group $Gal(\Omega_F^p/E_\infty)$ acts trivially on the abelian group $\mathbb{Q}_p/\mathbb{Z}_p(n)$, and therefore

$$Hom_{\mathbb{Z}_p}(\mathfrak{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) = H^1_{\text{ét}}(o'_{E_\infty}, \mathbb{Q}_p/\mathbb{Z}_p(n)),$$

where

$$H^1_{\text{ét}}(o'_{E_\infty}, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \varinjlim H^1_{\text{ét}}(o'_{E_m}, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

It is now easy to see that

$$H^1_{\text{ét}}(o'_{E_\infty}, \mathbb{Q}_p/\mathbb{Z}_p(n))^\Gamma = H^1_{\text{ét}}(o'_E, \mathbb{Q}_p/\mathbb{Z}_p(n)).$$

The parity of χ is equal to $(-1)^n$ and therefore by Corollary 2.2 the χ^{-1}-eigenspace of $H^1_{\text{ét}}(o'_E, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is finite, and isomorphic to $H^2_{\text{ét}}(o'_E, \mathbb{Z}_p(n))^{\chi^{-1}}$. We have shown:

Proposition 3.1. *The Pontryagin dual of $\mathfrak{X}^\psi(-n)_\Gamma$ is isomorphic to the finite group $H^2_{\text{ét}}(o'_E, \mathbb{Z}_p(n))^{\chi^{-1}}$.*

We now apply Lemma 1.1. Since $\mathfrak{X}^\psi(-n)$ has no non-trivial finite Λ-submodules, the Γ-invariants of $\mathfrak{X}^\psi(-n)$ are trivial, and therefore we can compute the order of $\mathfrak{X}^\psi(-n)_\Gamma$ in terms of the valuation of the characteristic polynomial at 0. By Lemma 1.2 the characteristic polynomial of $\mathfrak{X}^\psi(-n)$ is given by $f^*(\kappa(\gamma)^n(1+T)-1)$. Hence by the Main Conjecture:

Proposition 3.2.

$$|\mathfrak{X}^\psi(-n)_\Gamma| = |f(\kappa(\gamma)^n - 1)|_v{}^{-1} = |L_p(1-n, \psi)|_v{}^{-1} = |L(1-n, \chi)|_v{}^{-1},$$

provided that $\psi \neq 1$.

We can slightly reformulate the result: Let us write $a \sim_p b$ if the two rational numbers a, b have the same p-adic valuation. Let d_χ denote the degree of $\mathcal{O}_\chi$ over $\mathbb{Z}_p$. Then

$$L(1-n, \chi)^{d_\chi} \sim_p |H^2_{\text{ét}}(o'_E, \mathbb{Z}_p(n))^{\chi^{-1}}|,$$

provided that $\chi \neq \omega^n$.

If $\psi = 1$, then we have to take care of the polynomial $H_1(T)$ as well, and we can do a similar, but easier calculation for the Iwasawa module $X = \mathbb{Z}_p$ over E_∞, whose characteristic polynomial equals T. The result is that

$$|X(-n)_\Gamma|^{-1} = \kappa(\gamma)^n - 1,$$

that the dual of $X(-n)$ is equal to $\mathbb{Q}_p/\mathbb{Z}_p(n)^{\omega^n}$, and hence

$$|H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))^{\omega^n}| = \kappa(\gamma)^n - 1.$$

Since $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)$ is cyclic, there is only one eigenspace, hence we have $H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)^\chi \neq 1 \leftrightarrow \chi \neq \omega^n$. This finally leads to the main result in the semi-simple case:

Theorem 3.3. *Let χ be a 1-dimensional Artin character of order prime to p over a real field F. Then for any finite set S of primes of F containing S_p, and any $n \geq 2$, so that $\chi(-1) = (-1)^n$, we have*

$$L^S(1-n,\chi)^{d_\chi} \sim_p \frac{|H^2_{\acute{e}t}(o_E^S, \mathbb{Z}_p(n))^{\chi^{-1}}|}{|H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n)^{\chi^{-1}}|},$$

where E is any finite abelian extension of F of degree prime to p, containing F_χ.

Let us consider the special case that $\chi = 1$ and $n \geq 2$ is even. Recall that

$$H^2(o_F, \mathbb{Z}(n)) \cong \prod_p H^2_{\acute{e}t}(o'_F, \mathbb{Z}_p(n)).$$

Let us denote the order of $H^2(o_F, \mathbb{Z}(n))$ simply by $h_n(F)$ in analogy with the class number, and let us denote the order of $H^0(F, \mathbb{Q}/\mathbb{Z}(n))$ simply by $w_n(F)$. Then we obtain the following (for details at the prime 2 cf. [23]):

Theorem 3.4. *Let F be a totally real number field and let $n \geq 2$ be an even integer. Then*

$$\zeta_F(1-n) = \pm\frac{h_n(F)}{w_n(F)}.$$

up to multiples of 2

We remark that the 2-primary part of the Theorem is also true if F is abelian over $\mathbb{Q}$.

In the special case $n = 2$ this is the Birch-Tate Conjecture:

Birch-Tate Conjecture 3.5. *Let F be a totally real number field. Then*

$$\zeta_F(-1) = \pm\frac{|K_2(o_F)|)}{w_2(F)}$$

(up to possible multiples of 2 *if F is not abelian over* $\mathbb{Q}$*).*

Theorem 3.4 is a special case of the (cohomological version of the) Lichtenbaum Conjecture ([27]): Let F be an arbitrary number field, and let $n \geq 2$. Let $\zeta^*_F(1-n)$, the special value of ζ_F at $1-n$, denote the first non-vanishing coefficient in a Taylor expansion of the zeta-function $\zeta_F(s)$ around $1-n$.

Lichtenbaum Conjecture 3.6. *Up to powers of* 2*:*

$$\zeta_F^*(1-n) = \pm \frac{|K_{2n-2}(o_F)|}{|K_{2n-1}(o_F)_{tors}|} \cdot R_n^B(F).$$

Here $R_n^B(F)$ denotes the Borel regulator. If the Bloch-Kato Conjecture is true, then the Lichtenbaum Conjecture is true for abelian number fields (cf. [24, 25, 1, 20, 6]).

If we want to include the 2-primary parts into this conjecture, then we should replace the K-groups by motivic cohomology groups, i.e. we are led to the motivic reformulation:

Motivic Lichtenbaum Conjecture 3.7.

$$\zeta_F^*(1-n) = \pm \frac{|H^2_{\mathcal{M}}(o_F)|}{|H^1_{\mathcal{M}}(o_F, \mathbb{Z}(n))_{tors}|} \cdot R_n^{\mathcal{M}}(F).$$

Here $R_n^{\mathcal{M}}(F)$ is closely related to the Borel regulator. This conjecture is known to be true (assuming Bloch-Kato) if F is totally real abelian and $n \geq 2$ is even (cp. Theorem 3.4) and in a few other cases.

4 The Coates-Sinnott Conjecture

We now consider an arbitrary abelian extension E/F of number fields with Galois group G, and let S be a finite set of primes in F containing the primes ramified in E and the infinite primes. It is well-known that there exists a function $\theta^S_{E/F}(s)$ with values in the complex group ring $\mathbb{C}[G]$, such that

$$\chi(\theta^S_{E/F}(s)) = L^S_{E/F}(\chi^{-1}, s)$$

for all characters χ of G. We simply define

$$\theta^S_{E/F}(s) = \sum_{\chi} L^S_{E/F}(\chi^{-1}, s) e_\chi \in \mathbb{C}[G],$$

where –as before– the sum extends over all absolutely irreducible characters of G, and $e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi^{-1}(g) g$ denotes the idempotent belonging to χ. By a result of Klingen-Siegel $\theta^S_{E/F}(1-n)$ is contained in $\mathbb{Q}[G]$ for all $n \geq 1$, and it was shown by Deligne-Ribet that suitable multiples of $\theta^S_{E/F}(1-n)$ are actually contained in the integral group ring $\mathbb{Z}[G]$. More precisely

$$Ann_{\mathbb{Z}[G]}(H^0(E, \mathbb{Q}/\mathbb{Z}(n))) \cdot \theta^S_{E/F}(1-n) \subset \mathbb{Z}[G].$$

The ideal $Ann_{\mathbb{Z}[G]}(H^0(E,\mathbb{Q}/\mathbb{Z}(n))) \cdot \theta^S_{E/F}(1-n)$ is called the *n-th higher Stickelberger ideal* and denoted by $\mathrm{Stick}^S_{E/F}(n)$. The classical Stickelberger Theorem states that

$$\mathrm{Stick}^S_{E/\mathbf{Q}}(1) \subset Ann_{\mathbb{Z}[G]}(Cl(o_E)),$$

and Brumer conjectured that the same result holds for arbitrary abelian extensions E/F. For $n \geq 2$ another generalization of Stickelberger's theorem, involving higher Quillen K-groups, was suggested by Coates-Sinnott in the case $F = \mathbb{Q}$ and extended to arbitrary base fields by Sands and V. Snaith.

Coates-Sinnott Conjecture 4.1. *Let E/F be an abelian Galois extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subset Ann_{\mathbb{Z}[G]}(K_{2n-2}(o_E)).$$

We note that at negative integers $1-n$, $n \geq 2$, the Artin L-function $L^S_{E/F}(\chi, s)$ vanishes unless F is totally real and $\chi(-1) = (-1)^n$. Therefore one usually restricts attention to F totally real, and either E totally real and n even or E CM and n odd.

As before, the 2-primary information about this conjecture suggests that the K-groups should be replaced by motivic cohomology groups, i.e. the correct version should read

Motivic Coates-Sinnott Conjecture 4.2. *Let E/F be an abelian extension of number fields with Galois group G, and let $n \geq 2$. Then*

$$Stick^S_{E/F}(n) \subset Ann_{\mathbb{Z}[G]}(H^2_{\mathcal{M}}(o_E, \mathbb{Z}(n)))).$$

To approach the conjecture one considers each prime p separately, and shows that

$$Ann_{\mathbb{Z}_p[G]}(H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \theta^S_{E/F}(1-n) \subset Ann_{\mathbb{Z}_pG]}(H^2_{\text{ét}}(o^S_E, \mathbb{Z}_p(n)).$$

This gives the p-part of the cohomological version of the conjecture.

We want to show now that the Classical Main Conjecture implies the p-part (p odd) of the conjecture in the semi-simple case, i.e. we are considering an odd prime p, which does not divide the order of G. In the setting of the previous section (we are enlarging E to contain μ_p) we fix $n \geq 2$ and a character χ of G with parity $(-1)^n$, so that the character $\psi := \chi\omega^n$ is real. We first recall the definition and some of the properties of Fitting ideals.

The (first) *Fitting ideal* $Fitt_R(M)$ of a finitely generated R-module M is defined as follows: Choose a free resolution

$$R^m \xrightarrow{\beta} R^n \to M \to 0$$

of M. The Fitting ideal $Fitt_R(M)$ of M is the R-ideal generated by all $n \times n$-minors of the $n \times m$-matrix representing β. This definition is independent of the choice of the free resolution. One of the properties of the Fitting ideal is that it is contained in the annihilator of M:

$$Fitt_R(M) \subset Ann_R(M),$$

and the two ideals are equal if M is a cyclic R-module. It is now rather straightforward to compute the Fitting ideal of

$$\mathfrak{X}^S(-n)^\chi = \mathfrak{X}^{S,\psi}(-n).$$

This is a finitely generated torsion $\mathcal{O}_\chi[[T]]$- module without non-trivial finite submodules, and therefore by a result of Greither ([15], Theorem 2.2, [30], Lemma 2.3) has projective dimension ≤ 1. We emphasize that the proof does not need the module to be finitely generated as a $\mathbb{Z}_p$-module, hence one does not have to assume that the μ-invariant of $\mathfrak{X}^{S,\psi}(-n)$ is trivial.

Now, if M is a f.g. torsion R-module of projective dimension ≤ 1, then there is a resolution of M of the form

$$0 \to R^n \xrightarrow{\beta} R^n \to M \to 0,$$

hence

$$Fitt_R(M) = (det\, \beta)$$

is a principal ideal generated by the determinant of β.

In our case we have an injection

$$0 \to \mathfrak{X}^{S,\psi}(-n) \to \Lambda/(f_{\psi,S}(\kappa(\gamma)^n(1+T)-1))))$$

with finite cokernel, where as in section 3 $f_{\psi,S}(T)$ is a generating polynomial of the characteristic ideal of $\mathfrak{X}^{S,\psi}$. At all height 1 primes of Λ the two principal Fitting ideals $Fitt(\mathfrak{X}^S(-n)^\chi)$ and $(f_{\psi,S}(\kappa(\gamma)^n(1+T)-1))$ coincide, and it is then well known (*cf.* [19],Proposition 3.2.1) that this implies the equality of the two ideals. We obtain:

Proposition 4.3.

$$Fitt_{\Lambda}(X^S(-n)^{\chi}) = (f_{\psi,S}(\kappa(\gamma)^n(1+T)-1)).$$

As an immediate consequence we obtain a reformulation of the Classical Main Conjecture:

Corollary 4.4. *The Main Conjecture for ψ is equivalent to*

$$Fitt_{\Lambda}(X^{S,\psi}(-n)) = (G_{\psi,S}(\kappa(\gamma)^n(1+T)-1))$$

for all $n \geq 2$

We now descend to $X^{S,\psi}(-n)_{\Gamma}$. Its Fitting ideal over $\mathcal{O}_{\psi}$ is the image of $Fitt_{\Lambda}(X^{S,\psi}(-n)$ under the map $T \mapsto 0$, hence

Corollary 4.5.

$$Fitt_{\mathcal{O}_{\psi}}(X^{S,\psi}(-n)_{\Gamma}) = (L_p^S(1-n,\psi)),$$

if $\psi \neq 1$.

To treat the case $\psi = 1$ we consider the Iwasawa module $\mathbb{Z}_p(-n)$ and obtain

$$Fitt_{\mathbb{Z}_p}(\mathbb{Z}_p(-n)_{\Gamma}) = (\kappa(\gamma)^n - 1),$$

and therefore

Corollary 4.6.

$$Fitt_{\mathbb{Z}_p}(X^S(-n)_{\Gamma}) = Fitt_{\mathbb{Z}_p}(\mathbb{Z}_p(-n)) \cdot (L_p^S(1-n,1)).$$

For a finite module M the Fitting ideals of M and its dual M^*, and so we can dualize and finally take the sum over all eigenspaces to obtain

Theorem 4.7.

$$Fitt_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(o_E^S,\mathbb{Z}_p(n))) = Fitt_{\mathbb{Z}_p[G]}(H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \theta^S_{E/F}(1-n).$$

We note that this implies the p-part of the cohomological version of the Coates-Sinnott Conjecture, because the right-hand side equals $Ann_{\mathbb{Z}_p[G]}(H^0(E,\mathbb{Q}_p/\mathbb{Z}_p(n)\cdot \theta^S_{E/F}(1-n)$, which is then contained in $Ann_{\mathbb{Z}_p[G]}(H^2_{\acute{e}t}(o_E^S,\mathbb{Z}_p(n)))$. Since

$$H^2_{\acute{e}t}(o'_E,\mathbb{Z}_p(n)) \subset H^2_{\acute{e}t}(o_E^S,\mathbb{Z}_p(n))$$

we obtain

Corollary 4.8. *If $p \nmid |G|$, then the p-part of the cohomological version of the Coates-Sinnott Conjecture holds:*

$$Ann_{\mathbb{Z}_p[G]}(H^0(E, \mathbb{Q}_p/\mathbb{Z}_p(n))) \cdot \theta^S_{E/F}(1-n) \subset Ann_{\mathbb{Z}_pG]}(H^2_{ét}(o^S_E, \mathbb{Z}_p(n)).$$

To prove the cohomological version of the Coates-Sinnott Conjecture for all primes p, the Classical Main Conjecture has to be replaced by an Equivariant Main Conjecture. A version of this has been formulated and proven by Ritter-Weiss ([31]) under the hypothesis that the μ-invariant of the Iwasawa-module $\mathfrak{X}$ is trivial and p is odd. As a consequence Nguyen Quang Do proved the cohomological version of the Coates-Sinnott Conjecture (cp. [29]). Independently, this was also proven by Burns-Greither ([7]) under the same assumptions (and some additional restrictions on the primes p, if $F \neq \mathbb{Q}$) as a consequence of the Equivariant Tamagawa Number Conjecture. Most recently, Greither and Popescu gave a new and much more general approach to an Equivariant Main Conjecture, again under the assumption that the μ-invariant vanishes and that p is odd, which implies the Coates-Sinnott Conjecture, but gives many other results as well.

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