THE ETNC FOR DIRICHLET L-FUNCTIONS AT $s = 0$ - *DRAFT NOTES*

DAVID BURNS

Overview of lectures

The 'Equivariant Tamagawa Number Conjecture' (or 'ETNC' for short) was formulated by Flach and the present author in [BuF01] as a natural refinement of the 'Tamagawa Number Conjecture' of Bloch and Kato. The latter conjecture was originally formulated by Bloch and Kato in [BK] and then extended and refined by both Kato [K92, K93] and Fontaine and Perrin-Riou [FP]. This earlier seminal work of Bloch, Kato, Fontaine and Perrin-Riou was restricted to motives with commutative coefficients and also left unresolved an important 'sign ambiguity' problem. By combining Deligne's notion of virtual objects with the methods of relative algebraic K-theory the ETNC simultaneously extended the formalism of Tamagawa number conjectures to motives with non-commutative coefficients and also resolved the sign ambiguity problem (in the commutative case).

The ETNC formalism now underlies much of the important recent work in non-commutative Iwasawa theory (see, for example, Fukaya and Kato [FK] or Venjakob's survey article [V]). At the same time it has also provided a universal framework which, upon appropriate specialisation, has refined a very wide variety of well known and rather explicit conjectures and results concerning the leading terms at integer values of motivic L-functions ranging, for example, from the Rubin-Stark Conjecture to the conjectures of Chinburg in Galois module theory and the refinement of the Birch and Swinnerton-Dyer Conjecture formulated by Mazur and Tate. By now there is also an impressive amount of evidence, both theoretical and numerical and due to various authors, in support of important special cases of the ETNC (see, for example, Flach's survey article [Fl04] or [Bu10]).

Our main aim in these lectures is to review in as explicit a fashion as possible the special case of the ETNC that is relevant to the value at zero of Dirichlet L-functions, and hence also to the abelian case of Stark's Main Conjecture. We shall also explain some of the refinements of Stark's Main Conjecture that this case of the ETNC predicts.

1. Determinant modules

1.1. **Free modules.** Let R be a commutative unital noetherian ring. For each finitely generated free R-module M we set

$$[M]_R := \wedge_R^{\mathrm{rank}_R(M)}(M).$$

This is a free R-module of rank one: if one chooses an R-basis $\{m_i : 1 \le i \le d\}$ of M, then one obtains a (non-canonical) isomorphism of R-modules $[M]_R \xrightarrow{\sim} R$ by

Preliminary version of 14 July 2009.

sending $\wedge_{i=1}^{i=d} m_i$ to 1_R. If ϕ is a homomorphism of finitely generated free R-modules of the same rank d, then we write $[\phi]_R$ for the induced homomorphism of R-modules $[M]_R \to [N]_R$.

When R is clear from context we will often abbreviate $[M]_R$ to $[M]$. If N is a another finitely generated free R-module, then we will abbreviate the tensor product $[M]_R \otimes_R [N]_R$ to $[M]_R[N]_R$, or even just $[M][N]$.

We will use the following basic properties of this construction.

P1) $[0]_R = R$.

P2) If $\mathcal{E} : 0 \to M \to N \to P \to 0$ is an exact sequence of finitely generated free R-modules, then there is a canonical isomorphism of R-modules of the form $\iota(\mathcal{E}) : [N]_R \cong [M]_R[P]_R$.

P3) For each finitely generated free R-module M we set

$$[M]_R^{-1} := \mathrm{Hom}_R([M]_R, R),$$

regarded as an R-module via $r(\theta)(\delta) = \theta(r\delta)$ for each $r \in R, \theta \in [M]_R^{-1}$ and $\delta \in [M]_R$. Then there are canonical isomorphisms of R-modules of the form

$$\mathrm{ev}_M : [M]_R[M]_R^{-1} \xrightarrow{\sim} R, \;\; \delta \otimes \theta \mapsto \theta(\delta)$$

$$\mathrm{ev}_M : [M]_R^{-1}[M]_R \xrightarrow{\sim} R, \;\; \theta \otimes \delta \mapsto \theta(\delta).$$

P4) Each isomorphism of finitely generated free R-modules $\phi : M \to N$ gives rise to a canonical isomorphism of R-modules of the form

$$t(\phi) : [M]_R[N]_R^{-1} \xrightarrow{[\phi]_R \otimes \mathrm{id}} [N]_R[N]_R^{-1} \xrightarrow{\mathrm{ev}_N} R$$

$$t(\phi) : [M]_R^{-1}[N]_R \xrightarrow{[\phi]_R^{-1} \otimes \mathrm{id}} [N]_R^{-1}[N]_R \xrightarrow{\mathrm{ev}_N} R.$$

Remark 1.1. If $\mathcal{E}$ is an exact sequence as in P2), then the isomorphisms $\mathrm{ev}_M, \mathrm{ev}_N$ and ev_P allow one to regard $\iota(\mathcal{E})$ as an isomorphism $[M]_R[N]_R^{-1}[P]_R \xrightarrow{\sim} R$ or even $[M]_R^{-1}[N]_R[P]_R^{-1} \xrightarrow{\sim} R$. When applying isomorphisms of the form $\iota(\mathcal{E})$ we will usually not specify explicitly which one of these possible interpretations that we have in mind (believing that it will always be clear from context!)

Exercise 1.2. Describe $\iota(\mathcal{E})$ explicitly.

Exercise 1.3. What is the connection between the isomorphism $t(\phi)$ and the determinant of a matrix of ϕ (with respect to any choice of R-bases of M and N)?

1.2. **Semisimple algebras.** Let A be a finite dimensional semisimple commutative algebra. Then A decomposes as a finite product of fields $A = \prod_{i \in I} A_i$ and there is a corresponding decomposition of any finitely generated A-module M as a sum $M = \bigoplus_{i \in I} M_i$ where M_i is a finitely generated A_i-module. Since any finitely generated A_i-module is free we may therefore define an A-module by setting

$$[M]_A := \bigoplus_{i \in I} [M_i]_{A_i}.$$

If M is a free A-module, then this definition of $[M]_A$ agrees with that given in 1.1. In general, there is a (non-canonical) isomorphism of A-modules $[M]_A \cong A$ and the properties P1)-P4) in 1.1.1 (with R equal to each field A_i) combine to give analogous properties and isomorphisms $\iota(\mathcal{E}), \mathrm{ev}_M$ (where we set $[M]_A^{-1} := \mathrm{Hom}_A([M]_A, A)$) and $t(\phi)$ in this new setting.

1.3. **Orders.** For simplicity we only discuss determinants over the orders that are of most relevance for us. So we fix a finite abelian group G and write A for the group ring $\mathbb{Q}[G]$ (this is a finite dimensional semisimple commutative $\mathbb{Q}$-algebra) and $\mathfrak{A}$ for the subring $\mathbb{Z}[G]$ of A. For each prime p we write $\mathbb{Z}_{(p)}$ for the localisation of $\mathbb{Z}$ at p. For each $\mathbb{Z}$-module M we write $M_{(p)}$ for the localisation $M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

1.3.1. *Projective modules.* If M is finitely generated projective $\mathfrak{A}$-module, then a theorem of Swan (cf. [CuR, (32.1)]) implies that $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free A-module and further that for each prime p the localisation $M_{(p)}$ is a free $\mathfrak{A}_{(p)}$-module of rank $\mathrm{rank}_A(M_{\mathbb{Q}})$. We therefore obtain an $\mathfrak{A}_{(p)}$-submodule $[M_{(p)}]_{\mathfrak{A}_{(p)}}$ of $[M_{\mathbb{Q}}]_A$. It is possible to show that the intersection

$$[M]_{\mathfrak{A}} := \bigcap_p [M_{(p)}]_{\mathfrak{A}_{(p)}} \tag{1}$$

is a non-zero $\mathfrak{A}$-submodule of $[M_{\mathbb{Q}}]_A$. This construction has the following properties:

- If M is a finitely generated free $\mathfrak{A}$-module, then this definition coincides with that in 1.1.
- The construction of $[M]_{\mathfrak{A}}$ via localisations means that in general it is no longer true that there is an isomorphism of $\mathfrak{A}$-modules of the form $[M]_{\mathfrak{A}} \cong \mathfrak{A}$. However, for every prime p the localisation $([M]_{\mathfrak{A}})_{(p)}$ is equal to $[M_{(p)}]_{\mathfrak{A}_{(p)}}$ and so is isomorphic (non-canonically) to $\mathfrak{A}_{(p)}$. This means that $[M]_{\mathfrak{A}}$ is an 'invertible $\mathfrak{A}$-module', or a 'locally-free $\mathfrak{A}$-module of rank one'.
- There are natural analogues of the properties P1)-P4) in 1.1.1 and of the isomorphisms $\iota(\mathcal{E}), \mathrm{ev}_M$ (with $[M]_{\mathfrak{A}}^{-1} := \mathrm{Hom}_{\mathfrak{A}}([M]_{\mathfrak{A}}, \mathfrak{A})$) and $t(\phi)$ in this new setting.

M has finite p.d. if ∃ exact seq. $0 \to P_{-d} \to P_{-d+1} \to \cdots \to P_d \to M \to 0$ w/ P_i proj. This is equiv. to an exact seq. $0 \to P \to Q \to M \to 0$ w/ P, Q proj.

1.3.2. *Modules of finite projective dimension.* An $\mathfrak{A}$-module M has finite projective dimension if and only if for all subgroups J of G and all degrees i the Tate cohomology group $\hat{H}^i(J, M)$ vanishes (a condition which is automatically satisfied if, for example, M is a finite module of order prime to the order of G). If M is any finitely generated such module, then it is known that there exists an exact sequence of $\mathfrak{A}$-modules of the form

$$\mathcal{E} : 0 \to P \to N \to M \to 0$$

in which P and N are both finitely generated and projective.

Definition If M is a finitely generated $\mathfrak{A}$-module of finite projective dimension, then one chooses a resolution $\mathcal{E}$ as above and defines $[M]_{\mathfrak{A}}$ to be the image of the submodule $[P]_{\mathfrak{A}}^{-1}[N]_{\mathfrak{A}}$ of $[P_{\mathbb{Q}}]_A^{-1}[N_{\mathbb{Q}}]_A$ under the isomorphism $\iota(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{E})$: $[P_{\mathbb{Q}}]^{-1}[N_{\mathbb{Q}}]_A \cong [M_{\mathbb{Q}}]_A$.

This construction has the following properties:

- $[M]_{\mathfrak{A}}$ is independent of the choice of resolution $\mathcal{E}$.
- If M is a finitely generated projective $\mathfrak{A}$-module, then (M has finite projective dimension and) $[M]_{\mathfrak{A}}$ coincides with the module defined in 1.3.1.
- There are natural analogues of the properties P1)-P4) in 1.1.1 and of the isomorphisms $\iota(\mathcal{E}), \mathrm{ev}_M$ (with $[M]_{\mathfrak{A}}^{-1} := \mathrm{Hom}_{\mathfrak{A}}([M]_{\mathfrak{A}}, \mathfrak{A})$) and $t(\phi)$ in this new setting.

Exercise 1.4. Prove that $[M]_{\mathfrak{A}}$ is independent of the choice of $\mathcal{E}$.

If M is proj., take P = 0, N = M and so this agrees w/ 1.3.1

Remark: Best way to do this is via determinant functor due to Grothendieck – Knudsen – Mumford.

Exercise 1.5. Let M be a finite module. Show that M has finite projective dimension as a $\mathbb{Z}$-module and that $[M]_{\mathbb{Z}}$ is equal to the submodule $|M|^{-1}\cdot\mathbb{Z}$ of $\mathbb{Q} = [M_{\mathbb{Q}}]_{\mathbb{Q}}$. (since $M_{\mathbb{Q}} = 0$ b/c $M \otimes \mathbb{Q} = 0$)

2. Yoneda 2-extensions and determinant lattices

We fix a finite group G, set $\mathfrak{A} := \mathbb{Z}[G]$ and assume given $\mathfrak{A}$-modules M and N.

2.1. **Perfect 2-extensions.** . A '2-extension of M by N' is an exact sequence of $\mathfrak{A}$-modules of the form

$$\mathcal{T} : 0 \to N \to E_0 \xrightarrow{d} E_1 \to M \to 0.$$

If $\mathcal{T}'$ is another exact sequence $0 \to N \to E_0' \xrightarrow{d'} E_1' \to M \to 0$, then we write $\mathcal{T} \rightsquigarrow \mathcal{T}'$ if there exists a commutative diagram of $\mathfrak{A}$-modules of the form

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & N & \longrightarrow & E_0 & \xrightarrow{d} & E_1 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \phi^0} & & \downarrow{\scriptstyle \phi^1} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & E_0' & \xrightarrow{d'} & E_1' & \longrightarrow & M & \longrightarrow & 0. \end{array} \tag{2}$$

This relation $\rightsquigarrow$ fails to be an equivalence relation on the set of such 2-extensions (it is not in general symmetric) but nevertheless generates an equivalence relation. The associated set $\mathrm{Ext}^2_G(M,N)$ of equivalence classes is finite, of cardinality dividing a power of $|G|$, and has a natural (abelian) group structure. (For more details see, for example, [HS].)

Definition We say that an element ϵ of $\mathrm{Ext}^2_G(M,N)$ is *perfect* if it can be represented by an extension $0 \to N \to E_0 \to E_1 \to M \to 0$ in which the A-modules E_0 and E_1 are both finitely generated and of finite projective dimension.

Remark 2.1. The existence of a perfect element in $\mathrm{Ext}^2_G(M,N)$ is a strong restriction on the structures of M and N (for example, it immediately implies that they are both finitely generated) and is far from guaranteed.

2.2. **Determinant lattices.** We now assume to be given an isomorphism of $\mathbb{R}[G]$-modules

$$\lambda : N_{\mathbb{R}} \xrightarrow{\sim} M_{\mathbb{R}}.$$

We show that if ϵ is any perfect element of $\mathrm{Ext}^2_G(M,N)$, then one can associate to the pair (ϵ,λ) a canonical (invertible) $\mathfrak{A}$-submodule of $\mathbb{R}[G]$.

To do this we first choose a representative

$$\mathcal{T} : 0 \to N \to E_0 \xrightarrow{d} E_1 \to M \to 0$$

of ϵ. This exact sequence gives rise to short exact sequences of $\mathbb{R}[G]$-modules

$$\mathcal{E}_1 : 0 \to N_{\mathbb{R}} \to E_{0,\mathbb{R}} \to \mathrm{im}(d)_{\mathbb{R}} \to 0$$
$$\mathcal{E}_2 : 0 \to \mathrm{im}(d)_{\mathbb{R}} \to E_{1,\mathbb{R}} \to M_{\mathbb{R}} \to 0$$

and hence to a composite isomorphism of $\mathbb{R}[G]$-modules

$$\begin{aligned} \iota(\mathcal{T},\lambda) : [E_{0,\mathbb{R}}][E_{1,\mathbb{R}}]^{-1} &\xrightarrow{\iota(\mathcal{E}_1)\iota(\mathcal{E}_2)} ([N_{\mathbb{R}}][\mathrm{im}(d)_{\mathbb{R}}])([\mathrm{im}(d)_{\mathbb{R}}]^{-1}[M_{\mathbb{R}}]^{-1}) \\ &\xrightarrow{\mathrm{ev}_{\mathrm{im}(d)_{\mathbb{R}}}} [N_{\mathbb{R}}][M_{\mathbb{R}}]^{-1} \\ &\xrightarrow{t(\lambda)} \mathbb{R}[G]. \end{aligned}$$

Definition We set $\Xi(\epsilon, \lambda) := \iota(\mathcal{T}, \lambda)([E_0]_{\mathfrak{A}}[E_1]_{\mathfrak{A}}^{-1})$, where $[E_0]_{\mathfrak{A}}[E_1]_{\mathfrak{A}}^{-1}$ is regarded as a submodule of $[E_{0,\mathbb{R}}]_{\mathbb{R}[G]}[E_{1,\mathbb{R}}]_{\mathbb{R}[G]}^{-1}$ in the natural way.

The key result in respect of the above construction is the following.

Proposition 2.2. *$\Xi(\epsilon, \lambda)$ depends only upon ϵ and λ.*

Proof. We explain the key point. If ϵ is perfect then for any two representatives $\mathcal{T}$ and $\mathcal{T}'$ of ϵ one knows that $\mathcal{T} \rightsquigarrow \mathcal{T}'$. In particular, there exists a commutative diagram of the form (2). Consider the following related commutative diagram.

$$\begin{array}{ccccc} & & E_0' & \xrightarrow{d'} & E_1' \\ & & \downarrow{\scriptstyle (0,\mathrm{id})} & & \downarrow{\scriptstyle \mathrm{id}} \\ E_0 & \xrightarrow{(d,\phi^0)} & E_1 \oplus E_0' & \xrightarrow{(-\phi^1,d')} & E_1' \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \\ E_0 & \xrightarrow{d} & E_1. & & \end{array}$$

The central column is clearly a short exact sequence, call it $\mathcal{E}$, and it is also straightforward to show that the central row is a short exact sequence, call it $\mathcal{E}'$. The diagram therefore induces an isomorphism of $\mathfrak{A}$-modules

$$\begin{aligned} \iota(\mathcal{T}', \mathcal{T}) : [E_0'][E_1']^{-1} &\xrightarrow{t(\mathcal{E})} ([E_1 \oplus E_0'][E_1]^{-1})[E_1']^{-1} \\ &= ([E_1 \oplus E_0'][E_1']^{-1})[E_1]^{-1} \xrightarrow{t(\mathcal{E}')} [E_0][E_1]^{-1}. \end{aligned}$$

It can be shown that $\iota(\mathcal{T}', \lambda) = \iota(\mathcal{T}, \lambda) \circ (\mathbb{R} \otimes_{\mathbb{Z}} \iota(\mathcal{T}, \mathcal{T}'))$ and hence that

$$\iota(\mathcal{T}', \lambda)([E_0'][E_1']^{-1}) = \iota(\mathcal{T}, \lambda)(\iota(\mathcal{T}, \mathcal{T}')([E_0'][E_1']^{-1})) = \iota(\mathcal{T}, \lambda)([E_0][E_1]^{-1}).$$

□

Example If G is trivial, then the explicit computation of $\Xi(\epsilon, \lambda)$ is straightforward (see Exercise 2.3 below). To consider the simplest non-trivial case we assume that G is cyclic of prime order p and that $M = N = \mathbb{Z}$. In this case it can be shown that $\mathrm{Ext}^2_G(\mathbb{Z}, \mathbb{Z})$ has order p and moreover that every non-zero element of $\mathrm{Ext}^2_G(\mathbb{Z}, \mathbb{Z})$ is perfect. Further, if λ is any isomorphism of $\mathbb{R}[G]$-modules $\mathbb{R} \cong \mathbb{R}$, and ϵ and ϵ' any two non-zero elements of $\mathrm{Ext}^2_G(\mathbb{Z}, \mathbb{Z})$, then one has $\Xi(\epsilon, \lambda) = \Xi(\epsilon', \lambda)$ if and only if $\epsilon = \epsilon'$ (so $\Xi(\epsilon, \lambda)$ actually determines ϵ!).

Exercise 2.3. If G is the trivial group, then so is $\mathrm{Ext}^2_G(M, N)$ and if M and N are finitely generated the unique element 0 of $\mathrm{Ext}^2_G(M, N)$ is perfect. For any isomorphism λ of $\mathbb{R}$-modules compute explicitly the submodule $\Xi(0, \lambda)$ of $\mathbb{R}$. (Recall Exercise 1.5.)

Exercise 2.4. Prove that if λ' is any other isomorphism of $\mathbb{R}[G]$-modules, then $\Xi(\epsilon, \lambda') = \det_{\mathbb{R}[G]}(\lambda^{-1} \circ \lambda') \cdot \Xi(\epsilon, \lambda)$.

3. The conjecture

3.1. **Notation.** We now fix a finite abelian extension of global fields K/k and set $G := \mathrm{Gal}(K/k)$. We also fix a finite non-empty set of places S of k which contains all archimedean places (if any) and all places which ramify in K/k. We write $S(K)$ for the set of places of K which lie above those in S.

We write $\mathcal{O}_S$ for the ring of $S(K)$-integers in K and set $U_S := \mathcal{O}_S^\times, J_S := \prod_{w\in S(K)} K_w^\times$ and $C_S := J_S/\Delta(U_S)$, where Δ is the natural diagonal embedding $U_S \to J_S$. We write Y_S for the free abelian group on the set $S(L)$ and X_S for the kernel of the natural map $\epsilon : Y_S \to \mathbb{Z}$. We note that all of these groups are naturally G-modules.

3.2. **Canonical classes.** We recall the definition of certain canonical extension classes coming from class field theory.

3.2.1. We first assume S is large enough so that $\mathrm{Pic}(\mathcal{O}_S)$ vanishes and consider exact commutative diagram of $\mathbb{Z}[G]$-modules of the following form

$$\begin{array}{ccccccccccc} & & 0 & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U_S & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X_S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Delta} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\subset} & & \\ 0 & \longrightarrow & J_S & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & Y_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\epsilon} & & \\ 0 & \longrightarrow & C_S & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & 0. & & \end{array} \tag{3}$$

Lemma 3.1. *Assume that* $\mathrm{Pic}(\mathcal{O}_S)$ *vanishes. Then the extension classes of the second and third rows of the diagram (3) determine the extension class of the first row. Further, if the modules* A', B', A'' *and* B'' *have finite projective dimension, then the extension class of the first row is perfect.*

Proof. The key point in proving the first assertion is that if $\mathrm{Pic}(\mathcal{O}_S)$ vanishes, then C_S is the module of a class formation and so $H^1(G_v, C_S)$ vanishes for each place v in S. The proof of the second statement is standard homological algebra. □

Using important earlier work of Tate [T66], in [C] Chinburg proved the existence of a diagram (3) in which the second, resp. third, row is a representative of the semi-local, resp. global, canonical class of class field theory (for details of the canonical classes see, for example, [NSW]). Further, it was shown that in any such diagram the modules A', B', A'', B'' can be chosen to be of finite projective dimension. Following Lemma 3.1 we may therefore make the following definition.

Definition We let τ_S denote the perfect element of $\mathrm{Ext}^2_G(X_S, U_S)$ which is represented by the extension class of the first row of any diagram (3) in which the second, resp. third, row is a representative of the semi-local, resp. global, canonical class.

3.2.2. We now extend the definition of τ_S to the general case.

Lemma 3.2. *There exists a (finitely generated)* $\mathbb{Z}[G]$*-module* $\tilde{X}_S$ *which is unique up to unique isomorphism and has the following two properties:*

(i) *There is an exact sequence* $0 \to \mathrm{Pic}(\mathcal{O}_S) \to \tilde{X}_S \to X_S \to 0$.

(ii) *There exists a canonical perfect extension class τ_S in $\mathrm{Ext}^2_G(\tilde{X}_S, U_S)$ such that if $\mathrm{Pic}(\mathcal{O}_S)$ vanishes (so $\tilde{X}_S = X_S$), then τ_S coincides with the extension class defined in 3.2.1.*

Remark 3.3. The definition of τ_S in Lemma 3.2 makes clear the link to the more explicit description that is possible in the case that $\mathrm{Pic}(\mathcal{O}_S)$ vanishes, but is otherwise rather unsatisfactory. However, that's only because we are hiding its true origins. The natural way to define both the module $\tilde{X}_S$ and extension τ_S is in terms of the compactly supported (Weil-)étale cohomology of the constant sheaf $\mathbb{Z}$ on $\mathrm{Spec}(\mathcal{O}_S)$.

3.3. **Statement of the conjecture.** We set $G^\wedge := \mathrm{Hom}(G, \mathbb{C}^\times)$ and for each $\chi \in G^\wedge$ we write e_χ for the idempotent $\frac{1}{|G|}\sum_{g\in G}\chi(g^{-1})g$ of $\mathbb{C}[G]$ and $L_S(\chi, s)$ for the S-truncated Dirichlet L-function that is associated to χ. We then obtain a $\mathbb{C}[G]$-valued meromorphic function of a complex variable z by setting

$$\theta_S(z) := \sum_{\chi\in G^\wedge} L_S(\overline{\chi}, z)e_\chi. \tag{4}$$

It is straightforward to show (by using the algebra decomposition (5) below and the fact that complex conjugation is continuous) that the leading term $\theta^*_S(0)$ of $\theta_S(z)$ at $z = 0$ belongs to $\mathbb{R}[G] \subset \mathbb{C}[G]$.

The relevant case of the equivariant Tamagawa number conjecture predicts an explicit formula for the lattice $\mathbb{Z}[G] \cdot \theta^*_S(0) \subset \mathbb{R}[G]$. To state this we write

$$\lambda_S : U_{S,\mathbb{R}} \xrightarrow{\sim} \tilde{X}_{S,\mathbb{R}}$$

for the isomorphism of $\mathbb{R}[G]$-modules that is induced by (Lemma 3.2(i) and) the negative of the Dirichlet regulator map.

Conjecture 3.4. $\mathbb{Z}[G] \cdot \theta^*_S(0) = \Xi(\tau_S, \lambda_S)$.

Remark 3.5. The validity of Conjecture 3.4 is known to be independent of the choice of S and to behave functorially under change of extension (more precisely, if Conjecture 3.4 is valid for K/k then for each intermediate field E it is valid for both K/E and E/k). Conjecture 3.4 is also known to be valid unconditionally in each of the following special cases.

- K is a finite abelian extension of $\mathbb{Q}$. (The validity of Conjecture 3.4 in this case follows from results of Greither and the present author [BuG, Th. 8.1, Rem. 8.1] and later results of Flach dealing with the 2-primary part of the conjecture [Fl09].)
- There exists an imaginary quadratic field F which has class number 1 and is such that $F \subseteq k$, K/F is finite abelian and $[K : k]$ is both odd and divisible only by primes which split completely in $F/\mathbb{Q}$. (This case is proved by Bley in [Bl, Th. 4.2].)
- K/k is quadratic. (This case is proved by Kim in [Ki, §2.4, Rem. i)].)
- K is a global function field. (This case is proved in [BuLT].)

Remark 3.6. The formulation of Conjecture 3.4 looks nothing like anything that is formulated in [BuF01]! The main reason is that the case we are dealing with is very special: one can formulate a leading term conjecture using a single 2-extension of $\mathbb{Z}[G]$-modules (namely τ_S) unlike the general case in which one has to work with perfect complexes (rather than extensions) over $\mathbb{Z}_p[G]$ for each prime p and then piece things together in the style of the definition (1). The proof that Conjecture 3.4 is indeed a special case of the central conjecture of [BuF01] is in fact rather technical and is given in [BuF98].

Remark 3.7. If, as already alluded to in Remark 3.3, one reinterprets Conjecture 3.4 in terms of étale cohomology (by using the results of [BuF98]), then connections with other (commutative) conjectures become apparent. In particular, one finds that in the function field, resp. number field, case Conjecture 3.4 is an equivariant refinement of the relevant case of Lichtenbaum's conjecture [L, Conj. 8.1e)], resp. is a version without 'sign ambiguities' of the form discussed in [BuF01, Rem. 9] of the relevant case of Kato's earlier 'generalized Iwasawa main conjecture' [K93, Conj. 3.2.2]. (We note that, perhaps surprisingly, such sign ambiguities mean that the relevant case of [K93, Conj. 3.2.2] is itself insufficient to imply the refinements of Stark's Conjecture that we discuss in section 5.)

Exercise 3.8. Show that if $K = k$, then Conjecture 3.4 just recovers the analytic class number formula up to sign. (Recall Exercises 1.3 and 2.3.)

Exercise 3.9. Show that, in general, Conjecture 3.4 is sensitive to a change of sign of either τ_S or λ_S. (Recall Exercise 2.4.)

4. Stark's Conjecture

For each $\chi \in G^\wedge$ we write $\tilde{\chi} : \mathbb{C}[G] \to \mathbb{C}$ for the induced ring homomorphism. Then there is a natural identification of algebras

$$\mathbb{C}[G] \xrightarrow{\sim} \prod_{\chi \in G^\wedge} \mathbb{C}, \; x \mapsto (\tilde{\chi}(x))_\chi \tag{5}$$

and, with respect to this identification, one has

$$\mathbb{Q}[G] = \{(x_\chi)_\chi : (x_\chi)^\omega = x_{\chi^\omega} \text{ for all } \omega \in \mathrm{Aut}(\mathbb{C})\}. \tag{6}$$

For each homomorphism of $\mathbb{Q}[G]$-modules $\phi : U_{S,\mathbb{Q}} \to X_{S,\mathbb{Q}}$ we set

$$R(\phi) := \det_{\mathbb{R}[G]}(\lambda_S^{-1} \circ \phi_{\mathbb{R}}) \in \mathbb{R}[G]. \tag{7}$$

Proposition 4.1. *Conjecture 3.4 implies Stark's Main Conjecture.*

Proof. We assume the validity of Conjecture 3.4 and fix an isomorphism of $\mathbb{Q}[G]$-modules $\phi : U_{S,\mathbb{Q}} \to X_{S,\mathbb{Q}}$. Then $\phi^{-1}(X_{S,\mathbb{Q}}) = U_{S,\mathbb{Q}}$ and so one has

$$\begin{aligned}
\mathbb{Q}[G] \cdot \theta_S^*(0) &= \Xi(\tau_S, \lambda_S)_{\mathbb{Q}} \\
&= t(\lambda_S)([U_{S,\mathbb{Q}}][X_{S,\mathbb{Q}}]^{-1}) \\
&= \mathrm{ev}_{X_{S,\mathbb{Q}}}([\lambda_S(U_{S,\mathbb{Q}})][X_{S,\mathbb{Q}}]^{-1}) \\
&= \mathrm{ev}_{X_{S,\mathbb{Q}}}([\lambda_S \circ \phi^{-1}(X_{S,\mathbb{Q}})][X_{S,\mathbb{Q}}]^{-1}) \\
&= \mathrm{ev}_{X_{S,\mathbb{Q}}}([X_{S,\mathbb{Q}}][X_{S,\mathbb{Q}}]^{-1})\det_{\mathbb{R}[G]}(\lambda_S \circ \phi_{\mathbb{R}}^{-1}) \\
&= \mathrm{ev}_{X_{S,\mathbb{Q}}}([X_{S,\mathbb{Q}}][X_{S,\mathbb{Q}}]^{-1}) \cdot R(\phi)^{-1} \\
&= \mathbb{Q}[G] \cdot R(\phi)^{-1}.
\end{aligned}$$

It follows that $R(\phi)\theta_S^*(0)$ belongs to $\mathbb{Q}[G]$ and hence that

$$\tilde{\chi}(R(\phi)\theta_S^*(0))^\omega = \widetilde{\chi^\omega}(R(\phi)\theta_S^*(0)).$$

To deduce Stark's Main Conjecture (in the formulation given by Tate in [T84, Chap. I, Conj. 5.1]) it suffices to note that for each $\chi \in G^\wedge$ one has

$$\tilde{\chi}(R(\phi)\theta_S^*(0)) = \frac{L_S^*(\overline{\chi}, 0)}{\det_{\mathbb{C}}(\lambda_{S,\mathbb{C}} \circ \phi_{\mathbb{C}}^{-1} \mid e_\chi(X_{S,\mathbb{C}}))}. \tag{8}$$

□

Exercise 4.2. Prove (6) and (8).

5. Refinements of Stark's Conjecture

It has by now been shown that Conjecture 3.4 either recovers or implies refinements of a wide variety of more explicit (and often better known) conjectures, including each of the following:

- The 'Rubin-Stark Conjecture'.
- Popescu's Conjecture.
- The 'refined class number formula' of Gross.
- The 'refined class number formula' of Tate.
- The 'refined class number formula' of Aoki, Lee and Tan.
- The 'guess' formulated by Gross in [G, top of p. 195].
- The 'refined p-adic abelian Stark Conjecture' of Gross.
- Brumer's Conjecture.
- The Brumer-Stark Conjecture.
- The '$\Omega(3)$-Conjecture' of Chinburg.
- The 'Strong Stark Conjecture' of Chinburg.
- The 'Lifted Root Number Conjecture' of Gruenberg, Ritter and Weiss.

In this section we shall first discuss the connection between Conjecture 3.4 and the refined class number formulas of Gross, Tate et al, and then discuss a new refinement of Stark's Main Conjecture that in the spirit of a version of Brumer's Conjecture for derivatives of $\theta_S(z)$ at $z = 0$.

5.1. **Congruences for values.** We now discuss what Conjecture 3.4 predicts concerning the value

$$\theta_S(0) := \sum_{\chi \in G^\wedge} L_S(\overline{\chi}, 0) e_\chi$$

of the function $\theta_S(z)$ at $z = 0$.

We first need a preparatory lemma. We set $n := |S| - 1$.

Lemma 5.1. *(but see Remark 5.2 below!!) There exists a representative of the canonical class τ_S of the form*

$$0 \to U_S \to F \xrightarrow{\phi} F \xrightarrow{\pi} \tilde{X}_S \to 0 \tag{9}$$

where F is a finitely generated free G-module with a basis $\{b_i : 1 \le i \le d\}$ such that each of the following is true:

(i) $d \ge n$.
(ii) *The module $F_1 := \mathbb{Z}[G] \cdot \{b_i : 1 \le i \le n\}$ satisfies $F_1^G = \ker(\phi^G)$.*
(iii) *The module $F_2 := \mathbb{Z}[G] \cdot \{b_i : n < i \le d\}$ satisfies $\phi(F_2^G) \subseteq F_2^G$.*

Remark 5.2. Lemma 5.1 is clearly false as stated since the torsion subgroup of U_S is non-trivial and so there cannot exist an exact sequence of the form (9)! To correct the result one needs to replace U_S by a convenient approximating module which is torsion-free (for details see [Bu10]). But, in order not to hide the key ideas, we prefer to ignore this problem and simply state the 'result' as shown.

Exercise 5.3. Let ϵ be the extension class of (9). Choose sections ι_1 and ι_2 to the surjective $\mathbb{R}[G]$-module homomorphisms $F_\mathbb{R} \xrightarrow{\phi_\mathbb{R}} \mathrm{im}(\phi)_\mathbb{R}$ and $F_\mathbb{R} \xrightarrow{\pi_\mathbb{R}} X_{S,\mathbb{R}}$ respectively and write $\langle \phi, \lambda_S \rangle$ for the unique element of $\mathrm{Aut}_{\mathbb{R}[G]}(F_\mathbb{R})$ which is equal to $\iota_2 \circ \lambda_S$ on $U_{S,\mathbb{R}}$ and to $\phi_\mathbb{R}$ on $\iota_1(\mathrm{im}(\phi)_\mathbb{R})$. Prove that $\Xi(\epsilon, \lambda_S) = \mathbb{Z}[G]{\cdot}\mathrm{det}_{\mathbb{R}[G]}(\langle \phi, \lambda_S \rangle)$.

We return to consider $\theta_S(0)$. To do this we define an idempotent by setting

$$e_0 := \sum_\chi e_\chi$$

where χ runs over all elements of $G^\wedge$ with the property that $L_S(\overline{\chi}, 0) \neq 0$. Then one has

$$\begin{aligned}\theta_S(0) &= \theta_S(0)e_0 + \theta_S(0)(1-e_0)\\ &= \theta_S^*(0)e_0\\ &= u \cdot \mathrm{det}_{\mathbb{R}[G]}(\langle \phi, \lambda_S \rangle)e_0\\ &= u \cdot \mathrm{det}_{\mathbb{R}[G]}(\phi_\mathbb{R})e_0\\ &= u \cdot \mathrm{det}_{\mathbb{Z}[G]}(\phi) \qquad (10)\end{aligned}$$

where $u \in \mathbb{Z}[G]^\times$ and the second, resp. third, resp. fourth, resp. last, equality follows from the definition of e_0, resp. by combining Conjecture 3.4 with the 'result' of Lemma 5.1 and the formula of Exercise 5.3, resp. since $e_0(U_{S,\mathbb{Q}}) = 0$, resp. since $e_0(\ker(\phi)_\mathbb{Q}) = 0$. Now Lemma 5.1(ii) and (iii) imply that the matrix of ϕ with respect to the basis $\{b_i : 1 \leq i \leq d\}$ is a block matrix of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where $A \in \mathrm{M}_n(I_G)$ and $D \in \mathrm{M}_{d-n}(\mathbb{Z}[G])$ and all entries of the matrices B and C belong to I_G. It follows that $\det(A) \in I_G^n$ and hence that the equality (10) implies that

$$\theta_S(0) \equiv u \cdot \det(A)\det(D) \pmod{I_G^{n+1}}.$$

The key point now is to note that the right hand term here can be interpreted as the discriminant of a natural pairing of the form

$$\rho_\phi : \ker(\phi)^G \times \mathrm{Hom}_\mathbb{Z}(\mathrm{cok}(\phi)_G, \mathbb{Z}) \to I_G/I_G^2$$

(this fact is purely homological algebra) and so one has

$$\theta_S(0) \equiv \mathrm{disc}(\rho_\phi) \pmod{I_G^{n+1}}.$$

To deduce Gross's refined class number formula from here simply requires the following class field theoretic fact (for a proof of which see [Bu07]).

Theorem 5.4. *With respect to the natural identifications* $\ker(\phi)^G \cong \mathcal{O}_{k,S}^\times$ *and* $\mathrm{Hom}_\mathbb{Z}(\mathrm{cok}(\phi)_G, \mathbb{Z}) \cong X_{k,S}$ *and* $I_G/I_G^2 \cong G$ *the pairing* ρ_ϕ *coincides with the* G*-valued reciprocity pairing introduced by Gross in* [G]. *In particular, Conjecture 3.4 implies the refined class number formula of Gross* [G].

Remark 5.5. For the reason explained in Remark 5.2, the actual proof of Theorem 5.4 is a little more involved than that sketched above. However it is still true that much of the proof of Theorem 5.4 is homological algebra of a universal nature. Indeed, the same approach gives a natural analogue of Theorem 5.4 relating the refined Birch and Swinnerton-Dyer Conjecture of Mazur and Tate to the relevant case of the ETNC.

5.2. **A Brumer type conjecture for higher derivatives of L-series.** To be completed.

REFERENCES

[Bl] W. Bley, *On the equivariant Tamagawa number conjecture for abelian extensions of a quadratic imaginary field*, Doc. Math. **11** (2006) 73-118.

[BK] S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, In: 'The Grothendieck Festschrift' vol. 1, Progress in Math. **86**, Birkhäuser, Boston, (1990) 333-400.

[Bu07] D. Burns, *Congruences between derivatives of abelian L-functions at $s = 0$*, Inventiones Math. **169** (2007) 451-499.

[Bu10] D. Burns, *Leading terms and values of equivariant motivic L-functions*, to appear in Pure App. Math. Q. **6** (2010) (John Tate Special Issue, Part II).

[BuF98] D. Burns, M. Flach, *On Galois structure invariants associated to Tate motives*, Amer. J. Math. **120** (1998), 1343–1397.

[BuF01] D. Burns, M. Flach, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501–570.

[BuG] D. Burns, C. Greither, *On the Equivariant Tamagawa Number Conjecture for Tate motives*, Inventiones Math. **153** (2003) 303-359.

[BuLT] D. Burns, *Congruences between derivatives of geometric L-functions (with an appendix by D. Burns, K. F. Lai and K-S. Tan)*, submitted for publication.

[C] T. Chinburg, *Exact sequences and Galois module structure*, Ann. of Math. **121** (1985), 351–376.

[CuR] C. W. Curtis, I. Reiner, *Methods of Representation Theory, Vol. I and II,* John Wiley and Sons, New York, 1987.

[Fl04] M. Flach, *The equivariant Tamagawa number conjecture: a survey (with an appendix by C. Greither)*, Contemp. Math. **358** (2004) 79-126.

[Fl09] M. Flach, On the cyclotomic main conjecture for the prime 2, submitted for publication.

[FP] J.-M. Fontaine, B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, In: Motives (Seattle) Proc. Symp. Pure Math. **55**, I, (1994) 599-706.

[FK] T. Fukaya, K. Kato, *A formulation of conjectures on p-adic zeta functions in noncommutative Iwasawa theory*, Proc. St. Petersburg Math. Soc. Vol. XII, 1–85, Amer. Math. Soc. Transl. Ser. 2, **219**, Amer. Math. Soc., Providence, RI, 2006.

[G] B. H. Gross, *On the value of abelian L-functions at $s = 0$*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., **35** (1988) 177-197.

[GrRW] K. W. Gruenberg, J. Ritter, A. Weiss, *A local approach to Chinburg's root number conjecture*, Proc. London Math. Soc. **79** (1999), 47–80.

[HS] P. J. Hilton, U. Stammbach, *A course in homological algebra* (Springer-Verlag, New York, 1997).

[K92] K. Kato, *Iwasawa theory and p-adic Hodge theory*, Kodai Math. J. **16** no 1 (1993) 1-31.

[K93] K. Kato, *Lectures on the approach to Iwasawa theory of Hasse-Weil L-functions via B_{dR}, I*, in: *Arithmetic algebraic geometry (Trento, 1991)*, Lecture Notes in Math. **1553** (Springer, Berlin, 1993), 50–163.

[Ki] S. Y. Kim, *On the Equivariant Tamagawa Number Conjecture for Quaternion Fields*, Ph. D. thesis, King's College London, 2003.

[L] S. Lichtenbaum, *The Weil-étale topology on schemes over finite fields*, Compositio Math. **141** (2005) 689-702.

[MC] D. Macias-Castillo, *On higher order Stickelberger type theorems*, in preparation.

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields* (Springer-Verlag, Berlin, 2000).

[S] J-P. Serre, *Sur le résidue de la fonction zêta p-adique d'un corps de nombres*, C.R. Acad. Sci. Paris **278** (1978) 183-188.

[T66] J. Tate, *The cohomology groups of tori in finite Galois extensions of number fields*, Nagoya Math. J. **27** (1966), 709–719.

[T84] J. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$* (Birkhäuser, Boston, 1984).

[V] O. Venjakob, *From the Birch and Swinnerton-Dyer Conjecture to non-commutative Iwasawa theory via the Equivariant Tamagawa Number Conjecture - a survey*, in 'L-functions and Galois Representations', Proceedings of the 2004 Durham Symposium, pp. 333-380, Cambridge Univ. Press, 2007.

E-mail address: david.burns@kcl.ac.uk

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON WC2R 2LS, UNITED KINGDOM