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7-13-09


# Selmer groups for reducible representations:

$p$ prime
$F$ number field, (totally imaginary if $p=2$)
$S$ finite set of primes of $F$ including all primes over $p$.
$G_{F,S}$ = Galois group of maximal unramified outside of $S$ ext. of $F$.
$R$ = commutative local Noetherian ring with fin. res. field (ex. $R = \mathbb{Z}_p$).

Suppose we have a short exact seq. of $R[G_{F,S}]$-modules

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] & A \arrow[r] & A_2 \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

where $s$ is a splitting as $R$-modules.
Such an extension is defined by a cocycle:

$$\chi : G_{F,S} \longrightarrow \mathrm{Hom}_R(A_2, A_1)$$
$$\chi(\sigma)(a_2) = \sigma s(\sigma^{-1} a_2) - s(a_2).$$

$$\Psi_A^i : H^i(G_{F,S}, A_2) \longrightarrow H^{i+1}(G_{F,S}, A_2)$$
(cont. coh throughout)
coboundary.

**Lemma:** Let $f$ be an $i$-cocycle w/ values in $A_2$. Then
$$\Psi_A^i([f]) = [\chi \cup f].$$

**Proof for $i=1$:** $\Psi_A^1([f])$ is the class of $d(s \circ f)$.
As then

$$d(s\circ f)(\sigma,\tau) = \sigma s\, df(\tau) - s\circ f(\sigma\tau) + s\circ f(\sigma)$$
$$= \sigma s\circ f(\tau) - s(\sigma f(\tau))$$
$$+ s(\underbrace{\sigma f(\tau) - f(\sigma\tau) + f(\sigma)}_{df(\sigma\tau)\,=\,0})$$
$$= \chi(\sigma)(\sigma f(\tau))$$
$$= (\chi \cup f)(\sigma,\tau) \quad \blacksquare$$

<u>Standard example:</u> $F = \mathbb{Q}(\mu_p)$, $S = \{(1+\zeta_p)\}$, $R = \mathbb{Z}_p$

$U_F = \mathbb{Z}[\frac{1}{p}, \zeta_p]^\times \otimes \mathbb{Z}_p$ p-units

$A_F$ = p-part of class group of F.

$U_F \simeq H^1(G_{F,S}, \mathbb{Z}_p(1))$ (Uses Kummer theory

$A_F/pA_F \simeq H^2(G_{F,S}, \mu_p)$. or Poitou-Tate duality)

$a \in U_F \rightsquigarrow$ Kummer char. $\kappa_a(\sigma) = \sigma(a^{1/p})/a^{1/p}$.

$\kappa_a \in H^1(G_{F,S}, \mu_p)$, which gives an ext

$\rightsquigarrow$

$$0 \to \mu_p \to T_a \to \mathbb{Z}_p \to 0$$

Twist this seq:

$$\rightsquigarrow \quad 0 \to \mu_p^{\otimes 2} \to T_a(1) \to \mathbb{Z}_p(1) \to 0$$

$$\Psi^1_{T_a(1)} : \underset{U_F}{\underset{\wr\wr}{H^1(G_{F,S}, \mathbb{Z}_p(1))}} \longrightarrow \underset{A_F \otimes \mathbb{Z}/\mu_p.}{\underset{\wr\wr}{H^2(G_{F,S}, \mu_p^{\otimes 2})}}$$

As we have a cup product pairing on $H^1(G_{F,S}, \mathbb{Z}_p(1))$ to $H^2(G_{F,S}, \mu_p^{\otimes 2})$

$\rightsquigarrow$ ~~pairing~~ $(\ ,\ )_{p,F,S} : U_F \times U_F \to A_F \otimes \mu_p$.

$\Psi^1_{T_a(1)}(b) = (a,b)_{p,F,S} \quad \forall\, b \in U_F$.

<u>Conjecture (McCallum-S.):</u> The span of the image of $( \, , \, )_{p,F,S}$ is $A_F^- \otimes \mu_p$.

<u>Thm (S.):</u> $\Psi^1_{T(1)}$ is surj. for $p < 1000$.

Now back to the general setting, only w/ $R = \mathbb{Z}_p$.

$\mathfrak{X}_F$ = Galois group of max. abelian pro-$p$ unram. outside $S$-ext of $F$.

$\pi : G_{F,S} \twoheadrightarrow \mathfrak{X}_F$ by restriction. (this cocycle gives ext.)

$\rightsquigarrow$

$$0 \to \mathfrak{X}_F \to \mathcal{F} \to \mathbb{Z}_p \to 0$$

$\rightsquigarrow$

$$0 \to \mathfrak{X}_F(1) \to \mathcal{F}(1) \to \mathbb{Z}_p(1) \to 0.$$

$U_F$ = $p$-completion of $S$ units in $F$.

So we have

$$H^1(G_{F,S}, \mathbb{Z}_p(1)) \to H^2(G_{F,S}, \mathfrak{X}_F(1))$$
$$\cong H^2(G_{F,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathfrak{X}_F$$

the action on $\mathfrak{X}_F$ is trivial since action is conj. and $\mathfrak{X}_F$ is abelian.

$\Psi_F : U_F \to H^2(G_{F,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathfrak{X}_F$

$S$-reciprocity map

$Y_F$ = Gal. grp of the max. unram. ab. (pro) $p$ ext of

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$F$ in which all primes in $S$ split completely

$\underset{\text{CFT}}{\overset{\simeq}{\uparrow}}$ $A_F / \langle \{[\mathfrak{p}] : \mathfrak{p} \in S\} \rangle$

(Assume $A$ is either finitely or cofin. gen. over $\mathbb{R}$)

$$(*) \quad 0 \to Y_F \xrightarrow{2} H^2(G_{F,S}, \mathbb{Z}_p(1)) \to \bigoplus_{v \in S} \mathbb{Z}_p \to \mathbb{Z}_p \to 0.$$

<u>Lemma:</u> Let $A$ be a trivial $\mathbb{Z}_p[G_{F,S}]$-module.

Let $h \in H^1(G_{F,S}, A) \simeq \mathrm{Hom}_{cts}(G_{F,S}, A) \simeq \mathrm{Hom}(\mathfrak{X}_F, A)$

$\rightsquigarrow 0 \to A \to T_h \to \mathbb{Z}_p \to 0$.

Then

$$(\mathrm{id} \otimes h) \circ \Psi_F = \Psi^1_{T_h(1)} : U_F \to H^2(G_{F,S}, \mathbb{Z}_p(1)) \otimes A.$$

Thus, $\Psi_F$ interpolates the type of cup products considered earlier.

$$C^{\cdot}_c(G_{F,S}, A) = \mathrm{Cone}(C^{\cdot}(G_{F,S}, A) \xrightarrow{\ell_S} \bigoplus_v C^{\cdot}(G_{F_v}, A))[-1]$$

$$\text{"} \longrightarrow \bigoplus_{v \in S} H^{i-1}(G_{F_v}, A) \to H^i_c(G_{F,S}, A) \to H^i(G_{F,S}, A)$$

$$\longrightarrow \bigoplus_{v \in S} H^i(G_{F_v}, A) \to .$$

These can be nonzero only for $i = 1, 2,$ or $3$ because of the shift.

<u>Poitou-Tate duality:</u> Pontr. dual

$$H^i_c(G_{F,S}, A)^\vee \simeq H^{3-i}(G_{F,S}, A^\vee(1))$$

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**Selmer complex:**

$$0 \to A_1 \underset{r}{\leftarrow\!\!\to} A \underset{s}{\leftarrow\!\!\to} A_2 \to 0$$

$$C_f^{\cdot}(G_{F,S}, A) \to \mathrm{Cone}\left(C^{\cdot}(G_{F,S}, A) \xrightarrow{t_S} \bigoplus_{v \in S} C^{\cdot}(G_{F_v}, A_1)\right)[-1]$$

Assume $s$ is a splitting of $\mathbb{Z}[D_w]$-modules for some $w | v$ for each $v \in S$. (this is now needed)

$\Rightarrow \quad \chi|_{D_w} = 0$.

So locally $A$ looks like a direct sum of $A_1$ and $A_2$.

$$\to H_c^i(G_{F,S}, A_1) \to H_f^i(G_{F,S}, A) \to H^i(G_{F,S}, A_2)$$
$$\xrightarrow{\Theta_A^i} H_c^{i+2}(G_{F,S}, A_1) \to \cdots$$

**Fundamental extension:**

$$\chi : G_{F,S} \longrightarrow Y_F$$

$$\rightsquigarrow \quad 0 \to Y_F \to T \to \mathbb{Z}_p \to 0.$$

$$\Theta_F = \Theta_{T(1)}^1 : U_F \longrightarrow H_c^2(G_{F,S}, Y_F(1))$$
$$\simeq H_c^2(G_{F,S}, \mathbb{Z}_p(1)) \otimes Y_F$$
$$\simeq \mathrm{Hom}(\mathcal{X}_F, \mathbb{Q}_p/\mathbb{Z}_p)^{\vee} \otimes Y_F$$
$$\simeq \mathcal{X}_F \otimes_{\mathbb{Z}_p} Y_F$$

$$q_F = \Theta_{T(1)}^2 : H^2(G_{F,S}, \mathbb{Z}_p(1)) \longrightarrow H_c^3(G_{F,S}, Y_F(1))$$
$$\simeq Y_F.$$

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<u>Prop:</u> $q_F$ splits 2. (2 from (*))

<u>Thm:</u> The diagram

$$\begin{tikzcd}
 & H^2(G_{F,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} X_F \arrow[d, "sw \circ (q_F \otimes id)"] \\
U_F \arrow[ur, "\Psi_F"] \arrow[r, "\Theta_F"'] & X_F \otimes_{\mathbb{Z}_p} Y_F
\end{tikzcd}$$

switching maps

anticommutes.

<u>Cor:</u> Suppose $|S| = 1$. Then

$\Psi^1_{F(1)} : U_F \to Y_F \otimes_{\mathbb{Z}_p} Y_F$

has anti-symmetric image.

(i.e. look like sums of $a \otimes b - b \otimes a$)

<u>Application:</u> $F = \mathbb{Q}(\mu_p)$

$\Delta = Gal(F/\mathbb{Q})$

$M$ - $\mathbb{Z}_p$ - mod.

$M^{(i)} = \omega^i$ - eigenspace where $\omega$ = Teichmuller char.

Assume Vandiver's conj. i.e. $Y_F^+ = 0$.

$i$ odd

$$U_F^{(1-i)} = \langle \eta_i \rangle \qquad \eta_i = \prod_{\delta \in \Delta} (1 - \zeta_p^{\delta})^{\omega(\delta)^{i-1}}$$

$k$ even $\quad V_1^{\,-k} \neq 0 \iff p \mid B_k \quad (p,k)$ irregular

$2 \le k \le p-3$

**Thm:** Suppose $(p,\kappa)$ & $(p,\kappa')$ are irreg. Then

$$(\eta_{p-\kappa}, \eta_{\kappa+\kappa'-1})_{p,F,\zeta} \neq 0$$

$$\Longleftrightarrow (\eta_{p-\kappa'}, \eta_{\kappa+\kappa'-1})_{p,F,\zeta} \neq 0.$$

Mc.-S. : $(\eta_{p-\kappa}, \eta_{2\kappa-1})_{p,F,\zeta} = 0.$

Application: Under "mild" hypotheses, the Galois group of the maximal unram. pro-$p$ ext. of $\mathbb{Q}(\mu_{p^\infty})$ is abelian iff all of the values $(\eta_{p-\kappa}, \eta_{\kappa+\kappa'-1}) \neq 0$ for $\kappa \neq \kappa'$.