Refined class number formulas and Kolyvagin systems:

Gross conjecture: K/F finite abelian ext. w/ Galois group Γ.

S, T disjoint sets of places

$S \supset$ arch. places and places that ramify in K/F.

$\chi \in \hat{\Gamma}$, $L_{S,T}(\chi, s)$ as in previous lectures.

$\Theta_{S,T}(s) = \sum_{\chi \in \hat{\Gamma}} e_\chi L_{S,T}(\bar{\chi}, s)$.

$I < \mathbb{Z}[\Gamma]$ augmentation ideal, i.e.,

$I = \ker(\mathbb{Z}[\Gamma] \to \mathbb{Z} \; : \; \gamma \longmapsto 1)$

$I/I^2 \simeq \Gamma$

$\gamma - 1 \longleftrightarrow \gamma$

$r = \operatorname{rank} \mathcal{O}^\times_{F,S,T} = |S| - 1$.

Conjecture (Gross): If $w_{K,S,T} = 1$ (# of roots of unity of K congruent to 1 mod v $\forall v \in T$), then

1) $\Theta_{S,T}(0) \in I^r$

2) $\Theta_{S,T}(0) \equiv -h_{F,S,T} R_{F,S,T} \mod I^{r+1}$ where

$h_{F,S,T} = \#(Cl_{F,S,T})$ and

$R_{F,S,T} = \det(\underbrace{[\varepsilon_i, F_{w_j}] - 1}_{\in I/I^2})_{1 \le i,j \le r} \in I^r/I^{r+1}$ (Gross regulator)

Artin symbol $\in \Gamma$

- $\{\varepsilon_1, \ldots, \varepsilon_r\}$ is a $\mathbb{Z}$-basis of $\mathcal{O}^\times_{F,S,T}$
- $\{w_0, \ldots, w_{r+1}\} = S$
- $\det(\log|\varepsilon_i|_{w_j}) > 0$ (this condition fixes a sign)

This conjecture is known to be true if:

- K/F function fields Burns, Greither - Popescu
- $F = \mathbb{Q}$ Burns - Greither.
- if S contains a place that splits completely in K/F, $|S| \geq 2$.

Darmon's Conjecture:

Fix a real quadratic field $F/\mathbb{Q}$

$\omega = \omega_F$ corresp. Dirichlet char.

f = cond. of ω

$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ fixed embedding

$\zeta_m = e^{2\pi i/m}$

n sq. free integer prime to f

$K = F(\mu_n)$

$$\begin{tikzcd}[column sep=small] & & K \arrow[dl, dash] \arrow[d, dash, "\Gamma"] \\ & \mathbb{Q}(\mu_n) \arrow[d, dash] & F \arrow[dl, dash, "\langle\tau\rangle"] \\ & \mathbb{Q} & \end{tikzcd}$$

$S = \{\lambda : \lambda \mid fn\infty\}$

$T = \phi$

$\alpha = \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{fn})/\mathbb{Q}(\mu_n))} (\zeta_{nf}^{\sigma} - 1)^{\omega(\sigma)} \in (\mathcal{O}_K^{\times})^{-}$

$\Theta' = \sum_{\gamma \in \Gamma} \gamma\alpha \otimes \gamma \quad \in \quad K^{\times} \otimes \mathbb{Z}[G]$

Note: $\log|\cdot|_{v_\infty} \otimes \chi(\Theta') = -2 L_S'(\chi\omega, 0)$ with χ an even character

Thm (Darmon): $\Theta' \in (\mathcal{O}_K^{\times}/\{\pm 1\}) \otimes I^r$ where r is the rank of $(\mathcal{O}_{F,S}^{\times})^{-} - 1$, which is the same as the number of primes $\ell \mid n$, ℓ split in $F/\mathbb{Q}$.

Conj: $\Theta' \equiv -2^t h_{F,S} R_{F,n}$ in $(K^\times/\pm1) \otimes I^r/I^{r+1}$

where

$t = \#\{\ell \mid n,\ \ell \text{ not split in } F/\mathbb{Q}\}$

$R_{F,n}$ = Darmon regulator

$$= \det \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_r \\ \cdots & & \cdots \\ & [\varepsilon_i, F_{w_j}] - 1 & \end{pmatrix}_{(r-1)\times(r-1)} \in (\mathcal{O}_{F,S}^\times)^- \otimes I^r/I^{r+1}$$

$\{\varepsilon_1, \ldots, \varepsilon_r\}$ $\mathbb{Z}$-basis of $(1-\tau)\mathcal{O}_{F,S}^\times$

w_0 = chosen infinite place, $w_i w_i^\tau = \ell_i$, $\ell_1, \ldots, \ell_r$ prime divisors of n that split in F

$\cdot \det(\log|\varepsilon_i|_{w_j}) > 0$ (again fixes the sign)

n=1: $\Theta' = \alpha \in (\mathcal{O}_F^\times)^-$

$R_{F,1} = u/u^\tau$ where u is a fundamental unit of F.

where $|u|_{w_0} > 1$

So

$$-\tfrac{1}{2}\log|\alpha| = L'(\omega, 0) = h_F \log|u|_{w_0}$$
$$= \tfrac{1}{2} h_F \log|R_{F,1}|$$

$-\tfrac{1}{2}\log|\Theta'|$ (= $-\tfrac{1}{2}\log|\alpha|$)

Thus,

$$\Theta' = -h_F R_{F,1} \quad \text{in } \mathcal{O}_F^\times/\{\pm1\}$$

Thm (Mazur-R.): Darmon's conjecture is true up to the 2-part (true after $\otimes\, \mathbb{Z}[\tfrac{1}{2}]$)

Proof: $\mathcal{N} = \{$sq. free products of primes that split in $F\}$

Fix an odd prime p.

① LHS, RHS "almost" Kolyvagin systems for $\mathbb{Z}_p(1)\otimes\omega$

② $KS(\mathbb{Z}_p(1)\otimes\omega)$ free rk 1 / $\mathbb{Z}_p$.

LHS, RHS agree when $n=1$.

$\Rightarrow$ agree $\forall n \in \mathcal{N}$, $p \nmid n$.

$\Rightarrow$ agree $\forall n$.

$$\text{RHS} \in (F^\times)^- \otimes I_n^r/I_n^{r+1} \xrightarrow{\text{p-part}} H^1(\mathbb{Q}, \mu_{p^{\nu(n)}} \otimes \omega) \otimes I_n^r/I_n^{r+1}$$

($\nu(n)$ largest power of p dividing $\ell-1$ $\forall\ \ell \mid n$)

KS class $\in H^1(\mathbb{Q}, \mu_{p^{\nu(n)}}\otimes\omega)\otimes G_n$ $\qquad G_n = \bigotimes_{\ell \mid n} \Gamma_\ell$

$$\begin{array}{l} K_n = F(\mu_n) \\ \quad \Big| \ \Gamma_n \\ \quad F \\ \ / \\ \mathbb{Q} \end{array}$$

$\Gamma_\ell = Gal(F(\mu_\ell)/F)$

$\Gamma_n = \prod \Gamma_\ell$

$$\begin{array}{ccc} I_n/I_n^2 & \simeq & \bigoplus I_\ell/I_\ell^2 \\ \| & & \| \\ \Gamma_n & & \bigoplus \Gamma_\ell \end{array}$$

$G_n \longrightarrow I_n^r/I_n^{r+1}$ $\leftarrow$ generated by product $\prod_{i=1}^{r}(\gamma_i - 1)$ w/ $\gamma_i \in \Gamma_\ell$, some $\ell \mid n$ — monomial

$\gamma_{\ell_1}\otimes\cdots\otimes\gamma_{\ell_r} \mapsto \prod(\gamma_{\ell_i}-1)$

$\mathcal{C}(n)$ = cyclic subgroup of I_n^r/I_n^{r+1} given by

$\prod_{i=1}^{r}(\gamma_{\ell_i}-1)$, $\gamma_{\ell_i}\in\Gamma_{\ell_i}$, $\prod \ell_i = n$.

$\mathcal{J}(n)$ = subgrp gen. by all other monomials

Prop: ① $I_n^r / I_n^{r+1} = \mathcal{C}l(n) \otimes J(n)$

② $\mathcal{C}l(n) = \{ \rho \in I_n^r / I_n^{r+1} : \rho|_{\Gamma_{n/\ell}} = 0 \quad \forall \ell \mid n \}$

③ $G_n \xrightarrow{\sim} \mathcal{C}l(n)$ isom.

$$\gamma_1 \otimes \cdots \otimes \gamma_r \longmapsto \prod (\gamma_i - 1).$$

Pf: (Al Hales) □

KS(class $\in H^1(\mathbb{Q}, \mu_{p^{v(n)}} \otimes w) \otimes \mathcal{C}l(n)$.

LHS, RHS of conj. don't quite lie in

LHS: very closely related to Kolyvagin's derivative construction

of $d \mid n$,

$$\pi_d : \mathbb{Z}[\Gamma_n] \longrightarrow \mathbb{Z}[\Gamma_d] \hookrightarrow \mathbb{Z}[\Gamma_n]$$

$(1 \otimes \pi_{n/\ell})(R_n)$

$$\det \begin{pmatrix} \varepsilon_0, \ldots, \varepsilon_r \\ \pi_\ell[\varepsilon_j, F_{\ell j}] - 1 \\ 0 \cdots\cdots 0 \end{pmatrix} \qquad \begin{matrix} -h_{n/\ell}/h_n \\ \mathrm{Fr}_\ell^{\mathrm{ord}_\ell(\mathcal{E}_n)} - 1 \end{matrix}$$

$\varepsilon_0, \ldots, \varepsilon_{r-1}$ $\mathbb{Z}$-basis of n/ℓ ~~-units~~

$\ldots$ one gets $(1 \otimes \pi_{n/\ell})(R_n) = -\left(h_{n/\ell}/h_n\right) R_{n/\ell}$.

Define $\tilde{R}_n = \sum_{d \mid n} h_{n/d} R_{n/d} \prod_{\ell \mid d} \pi_{n/d}(\mathrm{Fr}_\ell - 1)$

Then

$$(1 \otimes \pi_{n/\ell})(\tilde{R}_n) = 0 \qquad \forall \ell$$

and so

$$\tilde{R}_n \in H^1(\mathbb{Q}, \mu_{p^{v(n)}} \otimes w) \otimes \underset{\substack{\shortparallel \\ G_n}}{\mathcal{C}l(n)}$$

One then needs a little more massaging to get them as Kolyvagin systems... ■