Non-optimal levels of reducible Galois representations:

$$\begin{array}{l}\bar{\mathbb{Q}} \\ \vert\,)G \\ \mathbb{Q}\end{array} \qquad \rho : G \longrightarrow \mathrm{Aut}\,V = GL_2(\mathbb{F}_{\ell^\nu}) \subseteq GL_2(\bar{\mathbb{F}}_\ell) \text{, representation}$$

- ρ odd ($\det \rho(\text{complex conj}) = -1$)
- ρ irreducible

These representations are said to be of "Serre type".

Conj: (1987): ρ is "modular"

Thm (Khare-Wintenberger): (2007): ρ is modular of level $N(\rho)$ and wt $k(\rho)$

$$\rho \longmapsto \begin{cases} k(\rho) \geq 2 & \text{weight } \rho|_{G_\ell} \text{ determines the weight} \\ N(\rho) \geq 1 & \text{level, prime to } \ell\text{, } p\text{-part depends on } \rho|_{G_p}. \end{cases}$$

wt k, level N, space of wt k cusp forms on $\Gamma_1(N)$. : $S_k(\Gamma_1(N))$
finite-dim vector space over $\mathbb{C}$, but has a rational structure.
Let $f \in S_k(\Gamma_1(N))$ and write $f(z) = \sum_{n \geq 1} a_n q^n$.
There is a family of Hecke operators acting on this space, they even preserve the rational structure.

$$\mathbb{T} = \mathbb{Z}[\{T_n\}] \subseteq \mathrm{End}\, S_k(\Gamma_1(N)).$$

$m = \ker\varphi \subseteq \mathbb{T} \xrightarrow{\varphi} \bar{\mathbb{F}}_\ell$, r prime $\quad \varphi(T_r) = \mathrm{trace}\,\rho(\mathrm{Frob}_r)$

is another way to say ρ is modular.

Another way to say ρ is modular is $\exists\, f \in S_k(\Gamma_1(N))$, $f \neq 0$, so that

$$f|T_n = a_n f,$$

$\mathbb{Z}[\{a_n\}] \subseteq \mathcal{O}_E \subseteq E$ is an order in a ring of integers $\mathcal{O}_E$, so giving a φ as above is the same as giving

$$m \subseteq \mathbb{Z}[\{a_n\}] \longrightarrow \overline{\mathbb{F}}_\ell$$
$$a_r \longmapsto \text{trace}\, \rho(\text{Frob}_r).$$

Even if one takes the minimal case, $(N = N(\rho),\ k = k(\rho))$, the "f" that makes ρ modular is not unique. One can have f be congruent to another form.

$N = N(\rho)$ optimal level
What are the possible M ~~with~~ $N|M$ s.t. $\exists$ f a newform of level M s.t. $f \rightsquigarrow \rho$??
Such an $M \neq N$ will be called a nonoptimal level. Moving from N to M is called "raising the level". The best papers on this are two papers by Diamond-Taylor. Basically, one can achieve all levels allowed by Galois invariants.

$k = 2$, $N = 11$, $\Gamma_0(11)$ $\quad S_2(\Gamma_0(11))$ has dim 1.
The fact we are using $k=2$ means we have the Jacobian of a modular curve: $J_0(11)$

$$J_0(11) \rightsquigarrow \rho \text{ of char } \ell$$

$$J_0(11)[\ell] : \rho : G \longrightarrow GL_2(\mathbb{F}_\ell).$$

ρ is irred if $\ell \neq 5$ and reducible if $\ell = 5$.

Take $\ell \neq 5$.

$$\rho : G \longrightarrow GL_2(\mathbb{F}_\ell).$$

This is modular of wt 2 and level 11 by construction, and these are optimal. What are non-optimal levels?

Raise level : level 11 $\rightsquigarrow$ $11p$, $p \neq 11, \ell$

$$S_2(\Gamma_0(11)) \rightsquigarrow S_2(\Gamma_0(11p)).$$

Nec. and sufficient condition on being able to raise the level and getting a newform: ratio of 2 eigenvalues of $\rho(\mathrm{Frob}_p)$ should be $p^{\pm 1}$ mod ℓ.

~~Trace~~ of $\rho(\mathrm{Frob}_p) = a_p \pmod{\ell}$

$$\det \rho(\mathrm{Frob}_p) = p \pmod{\ell}$$

This translates to

$$a_p \equiv \pm(1+p) \pmod{\ell}.$$

$a_2 = -2, \quad a_3 = -1, \quad a_5 = 1, \quad a_7 = -2, \ldots$

$a_7 - 8 = (-2) - 8) = -10$ 2, 5 are possible primes

$a_7 + 8 = 6$ 2 3 are possible primes

However, the rep is reducible at $\ell = 5$ so that prime doesn't count!

$$a_3 \pm 4 = \begin{cases} -5 \\ 3 \end{cases}$$

So this theory is pretty much known and settled in this case.

Now what about for ρ reducible? (semi-simple)
Since we are only interested in connecting via determinant and trace, it is enough to consider this case where we have the sum of two characters.

Eg: $1 \oplus \chi$ ← mod l cycl. char.

$J_0(11)[5] \simeq \mathbb{Z}/5\mathbb{Z} \oplus \mu_5$

The rep. $1 \oplus \chi$ arises from $J_0(p)$ whenever $p \equiv 1 \pmod{l}$ as long as $l \geq 5$.

Problem: Find those sq.-free N s.t. $1 \oplus \chi$ arises from (the new part of) $J_0(N)$. This was suggested by Deulefait.

Problem: Restrict attention to newforms f of wt 2 level $p_1 \cdots p_r$ (distinct primes) where r is fixed. Is it true that $\mathbb{Q}(\{a_n\})$ can be of arbitrarily high degree?

$r = 1$: $J_0(p)$

$f \rightsquigarrow 1 \oplus \chi$ (l large)

λ $\mathbb{Q}(\{a_n\})$ $\quad a_r \equiv 1 + r \pmod{\lambda}$

$|$

l

$$\boxed{a_2 \equiv 3 \pmod{\lambda}}$$

$$|a_2| \leq 2\sqrt{2} < 3$$

$a_2 - 3 \neq 0$, so $Nm(a_2 - 3) \neq 0$ and $l \mid Nm(a_2-3)$.

$\Rightarrow l \leq (3 + 2\sqrt{2})^{\deg(\mathbb{Q}(\{a_n\})/\mathbb{Q})}$

↑ multiplying conjugates to get norm..

$r=2$ can be done using reducible representations

<u>$r \geq 3$:</u> $\boxed{r=3}$ $3 \overset{\text{level raising}}{\rightsquigarrow} 4 \rightsquigarrow$

ℓ large prime

$$2^{\ell+4} = p_1 + p_2 p_3 \quad \text{(Chen 1970's)}$$

Assume there are 3. If only 2 (Goldbach), then do $r=2$ and use level raising.

$$\underset{A}{2^{\ell+4}} = \underset{B}{p_1} + \underset{C}{p_2 p_3}$$

Frey curve: E semi-stable

$$\operatorname{disc}(E) = \frac{(ABC)^2}{2^8} = 2^{2\ell}(p_1 p_2 p_3)^2$$

$E[\ell]$ irred $\rho : G \to GL_2(\mathbb{F}_\ell)$

level $p_1 p_2 p_3$

$a_2 \equiv \pm(1+2) \pmod{\ell}$

Now use the arg. from the $r=1$ case.