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Growth of L-values in towers:

I. Basic set-up:

$f = \sum_{n=1}^{\infty} a_n q^n$ cuspidal eigenform in $S_k(\Gamma_0(N))$ (normalized)

Fix p s.t. $p \nmid N$, $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Let $\mathbb{Q}_\infty$ = cyclotomic $\mathbb{Z}_p$-ext. of $\mathbb{Q}$. Fix a period $\Omega_f \in \mathbb{C}$ s.t.

$$\frac{L(1,f,\chi)}{\Omega_f} \in \bar{\mathbb{Q}},$$

for all finite order characters $\chi : \underbrace{\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})}_{=\Gamma} \to \mathbb{C}^\times$.

Question: How does the quantity

$$\mathrm{ord}_p\left(\tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}\right) \qquad (\mathrm{ord}_p(p) = 1)$$

vary as conductor of χ goes to ∞?

Remark: $f \longleftrightarrow E/\mathbb{Q}$ elliptic curve

$$\prod_{\chi : \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \to \mathbb{C}^\times} \tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f} \xleftrightarrow{BSD} |Ш(E/\mathbb{Q}_n)|\, \mathrm{Reg}(E/\mathbb{Q}_n).$$

grows like $p^{-\mathrm{rank}\, E(\mathbb{Q}_\infty)}$

II. Ordinary case:

Assume a_p is a p-adic unit. Then the p-adic L-function $\mathcal{L}_p(f) \in \mathcal{O}[[\Gamma]]$, $\mathcal{O}/\mathbb{Z}_p$ a finite extension. For each χ,

$$\chi(\mathcal{L}_p(f)) \sim \tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}.$$

up to p-adic unit

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There exist constants $\mu, \lambda \in \mathbb{Z}_{\geq 0}$ s.t.

$$\mathrm{ord}_p(\chi(\mathcal{L}_p(f))) = \mu/e + \frac{\lambda}{p^{n-1}(p-1)}$$, where order of χ is p^n and $n \gg 0$.

where e = ram of $\mathcal{O}/\mathbb{Z}_p$. $\mathcal{O}[[\Gamma]] \simeq \mathcal{O}[[T]]$

$\mathcal{L}(T) = a_0 + a_1 T + \cdots$

If ϖ is a uniformizer of $\mathcal{O}$, then $\mu = \min_i (\mathrm{ord}_{\varpi}\, a_i)$, λ is the smallest i s.t. $\varpi^{\mu} \| a_i$.

Remark: ① $f \longleftrightarrow E/\mathbb{Q}$,

$$\sum_{\chi : \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \to \mathbb{C}^\times} \mathrm{ord}_p\left(\tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}\right) = \mu p^n + \lambda n + O(1)$$

to take account of stuff for small n.

② Conjecture (Greenberg):

$\bar{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_p)$

If $\bar{\rho}_f$ is irreducible, then $\mu = 0$.

III Non-ordinary forms:

A) $f \longleftrightarrow E/\mathbb{Q}$

Thm: (Perrin-Riou / Kurihara): There exist constants $\mu^+, \mu^-, \lambda^+, \lambda^- \in \mathbb{Z}_{\geq 0}$ s.t.

$$\mathrm{ord}_p\left(\tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}\right) = \mu^{\pm} + \frac{q_n + \lambda^{\pm}}{p^{n-1}(p-1)}$$

where $q_n = p^{n-1} + p^{n-2} + \cdots + \left(\begin{matrix} p^2 - p \\ p^2 - 1 \end{matrix}\right.$ $\begin{matrix} \cancel{2\lambda n} \\ 21n \end{matrix}$

(slightly different if $|\mu^+ - \mu^-| = 1$).

Remark: ① Conjecture: $\mu^+ = \mu^- = 0$.
(E[p] is irred. as Galois module)

② There exist analogous algebraic formulas

B) General weight 2 (i.e. fourier coefficients not nec. in $\mathbb{Z}$)

Set $S_j = \frac{1}{p^j(p+1)}$. Let r be the smallest integer s.t. $S_r \leq \mathrm{ord}_p(a_p)$.

Theorem (Greenberg - Iovita - P.):

$$\mathrm{ord}_p\left(\tau(\bar{\chi})\frac{L(1,f,\chi)}{\Omega_f}\right) = \frac{\mu^\pm}{e} + r\cdot \mathrm{ord}_p(a_p) + \frac{q_{n-r} + \lambda^\pm}{p^{n-1}(p-1)}$$

(if $S_r \neq \mathrm{ord}_p(a_p)$.) For equality, there is a formula, just not giving it here.)

Remarks: ① When $\mathrm{ord}_p(a_p)$ is small, the growth is slower.

② There exist analogous algebraic formulas.

IV Rough sketch of proofs:

Mazur - Tate elements - These take the place of the p-adic

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L-functions from the ordinary case. They satisfy

$$\Theta_n(f) \in \mathcal{O}[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})]$$

s.t.

$$\chi(\Theta_n(f)) = \tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}.$$

corestriction

- $\mathrm{res}^n_{n-1}(\Theta_n(f)) = a_p\,\Theta_{n-1}(f) - \mathrm{cor}^{n-1}_{n-2}(\Theta_{n-2}(f)).$

restriction on group algebra

"Pf": $f \longleftrightarrow E/\mathbb{Q}$ $\quad p \mid a_p$

$$\mathrm{ord}_p(\chi(\Theta_n(f))) = \mathrm{ord}_p(\chi(\mathrm{cor}^{n-1}_{n-2}(\Theta_{n-2}(f))))$$
$$= \frac{p^{n-1}-p^{n-2}}{(p-1)p^{n-1}} + \mathrm{ord}_p(\chi(\Theta_{n-2}(f)))$$

.. continue doing this ... This gives the case A) of the theorem.

"Pf" of general case:

$$\mathrm{ord}_p(\chi(\Theta_n(f))) = \mathrm{ord}_p(a_p) + \mathrm{ord}_p(\Theta_{n-1}(f))$$

do three term relation on this ...

V Weights greater than 2:

$f \in S_k(\Gamma_0(N))$, ~~non-ord at~~ p.

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Still have Mazur-Tate elements:

$$\mathrm{res}(\Theta_n) = a_p \Theta_{n-1} - p^{k-2} \mathrm{cor}(\Theta_{n-2})$$

So if $k>2$, then the arg. before fails because p will divide both terms now!

Thm (P. - Weston): $f \in S_k(\Gamma_0(N))$

- $2 < k < p^2+1$ (medium weights)
- $\bar{\rho}_f$ irred. (no congruences to E.S.)
- $\bar{\rho}_f$ has Serre wt 2 and $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}$ non-split.

Then assuming the $\mu=0$ conjecture in wt 2,

① $\mu(\Theta_n(f)) = 0$ if $n \gg 0$

② $\lambda(\Theta_n(f)) = p^n - p^{n-1} + \begin{cases} \lambda & \bar{\rho}_f|_{G_{\mathbb{Q}_p}} \text{ is red.} \\ q_{n-1} + \lambda^{\pm} & \bar{\rho}_f|_{G_{\mathbb{Q}_p}} \text{ is irred.} \end{cases}$

Cor: If the $\{a_n\}$ generate an unramified ext of $\mathbb{Z}_p$, then

$$\mathrm{ord}_p\left(\tau(\bar{\chi}) \frac{L(1,f,\chi)}{\Omega_f}\right) = \frac{\lambda(\Theta_n(f))}{(p-1)p^{n-1}}.$$

Remarks: ① f congruent to a wt 2 form g. It is a fact g is ord iff $\bar{\rho}_g|_{G_{\mathbb{Q}_p}}$ is reducible.

② Only need $\mu=0$ for the form g.

$\lambda \longleftrightarrow \lambda(g)$

$\lambda^{\pm} \longleftrightarrow \lambda^{\mp}(g)$

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③ We show $\Theta_n(f) \equiv \mathrm{cor}(\Theta_{n-1}(g)) \pmod p$

④ There exist forms of wt. p^2+1 s.t. $\mu(\Theta_n) > 0$ for all n. The same is true for the restriction on the residual representation.

VI Proofs:

Assume for notational simplicity that $\mathcal{O} = \mathbb{Z}_p$.

Let

$$V_k = \mathrm{Sym}^{k-2}(\mathbb{Z}_p^2) \subseteq \mathbb{Z}_p[X,Y]$$

$$\bar{V}_k = \mathrm{Sym}^{k-2}(\mathbb{F}_p^2)$$

$$MS_\Gamma(V_k) = \mathrm{Hom}_\Gamma(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), V_k)$$

MS

$$\{r\} - \{s\}$$

r s

$$f \rightsquigarrow \varphi_f \in MS_\Gamma(V_k)$$

$$0 \neq \bar{\varphi}_f \in MS_\Gamma(\bar{V}_k)$$

$$\tilde{\Theta}_n(f) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \varphi_f\left(\left\{\tfrac{a}{p^n}\right\} - \{\infty\}\right)\Big|_{(0,1)} \sigma_n \in \mathbb{Z}_p\left[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})\right]$$

$$\rightsquigarrow \Theta_n(f).$$

Set $\Gamma = \Gamma_0(N)$, $\Gamma_0 = \Gamma_0(Np)$

$$\begin{aligned} MS_\Gamma(\bar{V}_k) &\xrightarrow{\alpha} MS_{\Gamma_0}(\mathbb{F}_p) \\ P(X,Y) &\longmapsto P(0,1) \\ \varphi_f &\longmapsto \alpha(\varphi_f). \end{aligned}$$

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Assume $\alpha(\varphi_f) \neq 0$.

& Ash-Stevens

get a cusp form $g \in S_2(\Gamma)$, $\bar{\rho}_g \simeq \bar{\rho}_f$

p-stabilization to go levels to match up.

$$\alpha(\bar{\varphi}_f) = \bar{\varphi}_g \Big|_{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)}$$

This gives that

$$\Theta_n(f) = \mathrm{cor}(\Theta_{n-1}(g)).$$

To see $\alpha(\varphi_f) \neq 0$: (know $k \equiv 2 \pmod{p-1}$).

$k = p+1$ Ash-Stevens α inj.

$k = 2p$ $\ker(\alpha) = MS_p(V_{p-1})(1)$

if $\alpha(\varphi_f) = 0 \Rightarrow g \in S_{p-1}(\Gamma)$

$\omega \otimes \bar{\rho}_g = \bar{\rho}_f$ not of wt 2.